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Decohering a Charged Scalar Field in a Time-Machine Wormhole Background

Malene Steen Nielsen Flagga[1]

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Is time travel possible? While this paper does not answer the question, it does put forward a model that may one day answer it. The decoherent-histories approach to quantum mechanics is used in a nontrivial background provided by a wormhole whose mouths reside in the same universe, but have a time difference between them. A charged scalar particle approaches the wormhole mouth in the present and is decohered spatially through the interaction with the Coulomb field of the wormhole mouth.

We start by reviewing the differential geometric setting for the matter fields, then discuss the wormhole metric and the time machine. Then we move on to the covariant derivates associated with the principal bundles erected to account for gauge and Lorentz transformations. Then the Lagrangian is discussed before we introduce the smeared position basis upon which the matter fields will be expanded. Then we come to the decoherent histories and the decoherence of the density matrix, and discuss how to handle evolution through causality-violating areas, as would be an area containing a time machine wormhole. Then we study the spatial decoherence of the matter field through the interaction with the charged wormhole mouth, in an attempt to localize the field before it would enter the wormhole. Finally, we discuss the effects the time machine has on such concepts as entropy and the direction of time.

1. THE MATTER FIELDS

Let M be a 4-dimensional manifold with charts $\{U_i\}$ and coordinate functions φ_i, with TM denoting the tangent bundle. We need a suitable

[1] The Niels Bohr Institute, University of Copenhagen, Copenhagen, Denmark; e-mail: manielsen@nbivms.nbi.dk.

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topology on our spacetime M of events $\phi^{-1}(x^{\mu})$. Let $I^{+}(m)$ denote the chronological future for m and $I^{-}(m)$ the chronological past. We then write $m \ll n$ to indicate that there exists a future-directed chronological curve from m to n [note that $m \notin I^{\pm}(m)$]. Inspired by the Alexandrov topology,[10] we define a basis for our topology as the set $\mathcal{U}$, consisting of the open subsets of M of the form

$$U_{i,x,y} = \{m \in M | n \ll m \ll r\} = I^{+}(n) \cap I^{-}(r)$$

We work with charged spin-0 particles. Had we chosen fermions, it would require a calculation of the first two Stiefel–Whitney classes in an environment described by the wormhole metric[11] which we introduce in the next section, a calculation which is beyond the scope of this article.[2] By working with spin-0 particles we avoid long and tedeous calculations and stick to the relevant physics. For our charged scalar matter fields, we erect a line bundle which is associated to both the frame bundle FM and to the principal bundle $P(M, U(1))$. These scalar fields are also sections $\psi: M \to L$ as $\psi(x) = \langle x | \psi \rangle \in L_m$. The equation

$$\psi(m) = \psi_m = \langle m | \psi \rangle \tag{1}$$

shows that $\langle m|$ is a functional on the set of sections, that is, $\langle m| : \Gamma(M, LM) \to \mathbb{C}$, although this is a complicated way of viewing M.

The line bundle $\Gamma(M, LM) \approx \mathcal{H} \approx \mathcal{L}_2(M)$ is the state-vector space for our scalar fields. It is obvious that the association of $\Gamma(M, LM)$ to $P(M, U(1))$ is the equivalence class saying that we identify two fields iff they only differ by a gauge. The association to $P(M, SO(3, 1))$ (the sections in the bundle of Lorentz transformations) is rather trivial since the scalar fields in $\Gamma(M, LM)$ carry the trivial representation of $SO(3, 1)$.

Since particles can appear and reappear as they travel through the wormhole, the need for second quantization is obvious. Both the scalar field and the electromagnetic potential will be expanded on complete sets of states, and the coefficients will be operators. In ordinary quantum mechanics the generalised coordinates are

$$\psi, \psi^* \in \Gamma(M, LM), \qquad \mathcal{A} \in \mathfrak{u}(1) \otimes \Omega^1(M)$$

and the configuration space is

$$\mathcal{C}(M) = \mathcal{T}^1(\Gamma(M, LM) \oplus \Omega^1(M) \otimes \mathfrak{u}(1))$$

Where the notation is that $\mathcal{T}^1(M)$ of a set M is a set of the functions on M and their first time derivatives. The phase space is a little tricky to construct, but it has the form

[2] It is a general requirement that for a manifold to admit a spin bundle, the second Stiefel–Whitney class must be trivial.

$$\mathcal{P}(\Sigma) = T^*(\Gamma(\Sigma, \mathbb{C}) \oplus \Omega^1(\Sigma) \otimes \mathfrak{u}(1))$$

where we have replaced M by the three-dimensional spacelike hypersurfaces Σ, expecting spacetime to be nontrivial. Here $\Omega^1(\Sigma)$ is the set of one-forms on Σ and $\mathfrak{u}(1)$ the Lie algebra belonging to the group Lie group $U(1)$.

To get the configuration space and phase space for the second-quantized fields, we must construct the Fock space for our theory, the vector space spanned by all possible states created by the creation and annihilation operators, $\hat{b}$, $\hat{b}^\dagger$ for the fields:

$$\mathcal{F}(\mathcal{H}) = \oplus_{n\in\mathbb{N}_0} \mathcal{H}^{\otimes n} = \oplus_{n\in\mathbb{N}_0} \mathcal{F}_n \tag{2}$$

(where we make the identification $\mathcal{H}^0 = \mathbb{C}$). Using

$$\hat{b}, \hat{b}^\dagger: \quad \mathcal{F}_n \rightarrow \mathcal{F}_{n\pm1} \tag{3}$$

we have

$$\hat{b}, \hat{b}^\dagger \in \oplus_{n\in\mathbb{N}_0}(\bar{\mathcal{F}}_{n\pm1} \otimes \mathcal{F}_n) \subset \mathcal{F}(\mathcal{H})^\dagger \otimes \mathcal{F}(\mathcal{H}) \tag{4}$$

So in going from classical variables to quantum variables (or second-quantizing), the configuration space is changed from $\mathcal{C}(M)$ to $\mathcal{F}(\mathcal{H})$.

The same result could have been achieved by starting with $\Gamma(M, LM)$. Since $\psi \in \Gamma(M, LM)$, when second-quantizing this field we get an operator-valued section:

$$\hat{\psi} \in \hat{\Gamma}(M, LM) = \mathcal{B} \otimes \Gamma(M, LM) \tag{5}$$

where we have introduced the algebra $\mathcal{B}$ as a set of operators spanned by the creation and annihilation operators. In our curved spacetime we shall use $\mathcal{B}(U_i) = span\{\hat{b}^\dagger(f), \hat{b}(f) \big| f \in C_c^\infty(U_i)\}$. The last term in Eq. (5) can be written

$$\begin{aligned}\mathcal{B}(\Gamma(M, LM)) &= \oplus_{n=0}^{\infty} \mathcal{B}^{\otimes n}(\Gamma(M, LM)) \\ &= \mathcal{B}\Gamma(M, LM) + \mathcal{B}\mathcal{B}\Gamma(M, LM) + \dots \\ &= \mathcal{F}(\mathcal{H})\end{aligned} \tag{6}$$

An important thing in Eq. (4) is that we have already taken into account the possible subspaces of $\mathcal{H}$ generated by an observable. Commonly $\hat{b}$ and $\hat{b}^\dagger$ are encountered as, e.g., $\hat{b}(k)$ and $\hat{b}^\dagger(k')$, when momentum or plane wave states are chosen. This splits the Hilbert space into subspaces, each characterized by the momentum k of the plane wave. Other choices for the basis upon which we expand our matter field are possible, and here we shall use a "smeared" position basis.

What we have constructed is an operator space $\mathcal{F}(\mathcal{H}) = \bigoplus_{n \in \mathbb{N}_0} \mathcal{H}^{\otimes n}$ over our Hilbert space $\mathcal{H}$ and introduced operators $\hat{b}^\dagger$ and $\hat{b}$ that act on the Hilbert space and satisfy

$$[\hat{b}(g), \hat{b}^\dagger(f)] = \langle g | f \rangle, \qquad f, g \in C_c^\infty(U_i) \tag{7}$$

where the demands that f, g be infinitely differentiable and have compact support ensure that the inner product exists in $\mathcal{H}$. If we take, for example, $f = e^{-ikx}$ and $g = e^{-ik'x}$ in the momentum space representation, we find

$$[\hat{b}(\vec{k}), \hat{b}^\dagger(\vec{k}')] = \langle e^{ikx} | e^{-ik'x} \rangle = \delta_{\vec{k}\vec{k}'} \tag{8}$$

which is a well-known result.

2. THE WORMHOLE

We now come to the introduction of the wormhole in our spacetime manifold M and the effect this has on the open coverings. Two things must be taken into consideration:

- The metric must accurately describe a wormhole and be free of event horizons.
- Away from the wormhole throat, spacetime must tend to asymptotic simplicity. That is, away from the wormhole we should recover a flat spacetime.

2.1. Definitions and Energy Conditions

Definition 1. If a Lorentzian spacetime contains a compact region Ω, and if the topology of Ω is of the form $\Omega = \mathbb{R} \times \Sigma$, where Σ is a 3-manifold of *nontrivial* topology, whose boundary has topology of the form $\partial\Sigma \sim S^2$, and if furthermore the hypersurfaces Σ are all spacelike, then the region Ω contains a quasi-permanent intra-universe wormhole.[(13),3]

For the wormhole to be *traversable* we must further demand that it contain no event horizon, which amounts to demanding that there must be no curvature singularities in the Riemann tensor (which is indeed the case

[3] By *intra-universe* is meant a wormhole that connects two regions of the same universe, as opposed to an *inter-universe* wormhole, which connects two different universes. Wormholes can also be *permanent, quasi-permanent*, or *transient.* If we slice a region of spacetime into spacelike hypersurfaces and each slice of space contains a wormhole, then the wormhole can be thought of as existing throughout a certain duration of time. These are quasi-permanent wormholes. A wormhole that is considered to exist throughout the lifetime of the universe is permanent, while transient wormholes pop into and out of existence without not even locally having a topologically structure of the form $\Omega \simeq \mathbb{R} \times \Sigma$. Transient wormholes are intrinsically four-dimensional objects.

with the metric with which we shall work[5]) and that the Riemann tensor is not too large. Since we work with a charged scalar field, it can withstand rather harsh tidal forces, but too large a Riemann tensor may, in a sense, rip the field apart or delocalize it.

2.2. Metric Considerations

Wormhole geometry is best approached using the *Wheeler–Thorne wormhole*,[13]

$$ds^2 = e^{2\Phi(l)}\, dt^2 - dl^2 - r^2(l)[d\theta^2 + \sin^2\theta\, d^2\phi] \tag{9}$$

where the variable l is the proper radial distance and $r(l)$ is the radius of spherical shells surrounding the wormhole. An observer A placed at a safe[4] distance will measure the distance l to the "center" of the wormhole, while an observer B moving toward this center will measure the radial distance $r(l)$. As long as B is sufficiently far away from the geometrically nonsimple area, he will measure the same distance to the center as A. This is expressed as

$$\lim_{l\to\pm\infty} \frac{r(l)}{|l|} = 1 \tag{10}$$

As B approaches the "center" of the wormhole he will observe his radial distance to the center decreasing until he reaches min $r(l) = r_0$, the radius of the wormhole throat. Even though B still feels as if he is moving on a straight line, his radial distance to the "center" will then increase again. He has completed a trip through the wormhole.

The function $\Phi(l)$ is known as the *redshift function*, and it is easily seen that the redshift function tells us how time and proper time are related.

We could then proceed and calculate the Riemann tensor and solve the Einstein equations, but doing our calculations using (9) is not an easy task. It is easier first to work in a combined Schwarzchild–Reissner–Nordström geometry (since we also want the wormhole mouth to be charged), and then simply reparametrize the functional dependence of the metric, meaning that instead of having the metric as a functional of l, we have it as a functional of $r = r(l)$. This is done by using $dl/dr = 1/[1 - b(r)/r]$, where $b(r)$ is known as the *shape function*:

$$ds^2 = e^{2\Phi_\pm(r)}\, dt^2 - \frac{dr^2}{1 - b_\pm(r)/r} - r^2[d\theta^2 + \sin^2\theta\, d^2\phi] \tag{11}$$

where the proper radial distance l is related to the radial distance r as

[4] By "safe" I mean in the asymptotically flat region of spacetime (see Section).

$$l(r) = \pm \int_{r_0}^{r} \frac{dr'}{\sqrt{1 - b_\pm(r')/r'}}$$

- The proper radial distance l covers the range $[-\infty, +\infty]$, while if we work with the r coordinate we exchange this for *two* coordinate patches each covering the range $[r_0, \infty]$.
- We have replaced the two functions $\Phi(l)$ and $r(l)$ in ref. 9 with the four functions $\Phi_+(r)$, $\Phi_-(r)$, $b_+(r)$, and $b_-(r)$; this is not an increase in the number of parameters, since their domains have been halved. We use $\Phi_+(r)$ and $b_+(r)$ when we are on one side of the wormhole (the + side), and $\Phi_-(r)$ and $b_-(r)$ when we are on the other side (the − side).[5]

In his book on wormholes, Visser[(13)] moves on to introduce certain constraints on the redshift and shape functions to ensure that the wormhole is traversable in principle. These restrictions are met with[(5)]

$$b_\pm(r) \equiv 2GM_\pm - \frac{Q_\pm^2}{r} \tag{12}$$

$$\Phi_\pm(r) \equiv \log\left(1 - \frac{b_\pm(r)}{r}\right) \tag{13}$$

since

$$\frac{b_\pm(r)}{r} \to 0 \qquad \text{for} \quad r \to \infty$$

$$\Rightarrow \log\left(1 - \frac{b_\pm(r)}{r}\right) \to 0 \qquad \text{for} \quad r \to \infty \tag{14}$$

$$\Rightarrow 2\Phi_\pm(r) \to 0 \qquad \text{for} \quad r \to \infty \tag{15}$$

$$\Rightarrow e^{2\Phi_\pm(r)} \to 1 \qquad \text{for} \quad r \to \infty$$

The difference between being “traversable in practice” as opposed to “traversable in principle” is whether or not one includes tidal effects for the wormhole. This can be done by writing down two inequalities. The first of these inequalities is a constraint on the gradient of the redshift function $\Phi(r)$, while the second effectively is a constraint on the speed with which the traveling object can safely traverse the wormhole.[(■)]

If we look at the metric we have used thus far, it is not hard to construct a time machine; it actually appears as if it is the generic fate of wormholes

[5] When we later turn the wormhole into a time machine, the + side will represent the wormhole mouth in the present, and the − side the wormhole mouth in the past.

that they evolve into time machines.[13] The quick argument is simply that since the two mouths of the wormhole reside in "different" parts of the universe, it is not hard to imagine that a certain acceleration takes place between the relative positions of the mouths, thus leading to a time shift.

3. A NOTE ON THE COVARIANT DERIVATIVES

It is important to understand the connection $\mathcal{A}_{i\mu}$ for the $U(1)$ principal bundle here since it gives rise to a change in the derivatives as

$$\partial_\mu \to D_\mu = \partial_\mu + \rho(A^a_\mu T_a) \tag{16}$$

where ρ is the representation of the transformation group. In our case we have *two* transformations groups: $U(1)$ for the gauge transformations and $SO(3, 1)$ for the Lorenz transformations. The associated bundle containing the matter field thus shares transition functions with both the tangent bundle, viewed as the principal bundle $P(M, SO(3, 1))$, and the $U(1)$ principal bundle. In QED it changes the derivative to

$$\partial_\mu \to D_\mu = \partial_\mu - ieA_\mu$$

with $A = A_\mu dx^\mu$, while if we couple to a gravitational field, we have

$$\partial_\mu \to \partial_\mu + \Gamma^a_{\mu b} T^b_{\ a}$$

with $A = \Gamma^a_{\mu b}\, dx^\mu T^b_{\ a}$. It is evident here that the roles played by the indices a, b, and μ are very different: μ is the $\Omega^1(M)$ index, while a and b are $\mathcal{SO}(3, 1)$ indices.

We thus have

$$D \in TM \otimes U(1) \otimes SO(3, 1) \otimes \mathcal{D} \tag{17}$$

where $\mathcal{D}$ denotes the set of derivatives. Since we work in a background consisting of both the gravitational field $g_{\mu\nu}$ and an electromagnetic field A_μ, the derivatives should in general be changed to covariant derivates taking the two different connections into account. However, the scalar matter fields carry the trivial representation of the Lorentz group $SO(3, 1)$ and thus it is only the electromagnetic connection that will appear in the covariant derivative. This would *not* have been the case had the fields been spinors.

Since the wormhole is charged, there are two "kinds" of electromagnetic potentials. One, which we shall term $A^{cl}_\mu(x)$, is the electric background field from the wormhole, the Coulomb field, and A^g_μ, is the electromagnetic quantum field, or vacuum fluctuations, responsible for creating and annihilating photons. The electromagnetic background field is stationary, so we have $A^{cl}_\mu = (A^{cl}_0, \vec{0}) = A^{cl}_0$,

$$A_\mu(x) = A_\mu^{cl} + A_\mu^{g} \tag{18}$$

giving rise to a covariant derivative of the form

$$\tilde{D}_\mu = \partial_\mu - ie(A_\mu^{cl} + A_\mu^{q}) \tag{19}$$

The classical field simply gives rise to corrections in the Hamiltonian, in effect raising the zero-point energy.

Discarding second-order and higher terms in the electromagnetic potential, we now have the Lagrangian

$$\begin{aligned}\mathcal{L}_{\psi,0}x) &= \sqrt{-g}\,[g^{\mu\nu}\tilde{D}_\mu^*\psi^*(x)\tilde{D}_\nu\psi(x) - (m^2 + \xi R)\psi^*(x)\psi(x)]\\ &= \sqrt{-g}\,[g^{\mu\nu}(\partial_\mu + ieA_\mu(x))\psi^*(x)(\partial_\nu - ieA_\nu(x))\psi(x)\\ &\quad - (m^2 + \xi R)\psi^*(x)\psi(x)]\end{aligned} \tag{20}$$

We also need to take into account the electromagnetic photon field. This is done, as usual, by adding the term

$$\begin{aligned}\mathcal{L}_{em}(x) &= \sqrt{-g}\left[-\frac{1}{4}F_{\mu\nu}(x)F_{\mu\nu}(x)\right]\\ &= \sqrt{-g}\,\frac{1}{4}\,[2F_{0i}(x)F^{0i}(x - F_{ij}(x)F^{ij}(x)]\end{aligned} \tag{21}$$

giving us the full Lagrangian for the theory as

$$\begin{aligned}\mathcal{L}(x) = \sqrt{-g}\Bigg[& g^{\mu\nu}(\partial_\mu - ieA_\mu(x))\psi^*(x)(\partial_\nu + ieA_\nu(x))\psi(x)\\ & -(m^2 + \xi R)\psi^*(x)\psi(x) - \frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x)\Bigg]\end{aligned} \tag{22}$$

The full action for the theory

$$\begin{aligned}S[\psi, \partial\psi] &= \int dt\, d^3x\, \sqrt{g}\,\mathcal{L}(\psi(x), \partial\psi(x))\\ &= \int d^4x\,\sqrt{g}\Bigg[\frac{1}{2}g^{\mu\nu}\tilde{D}_\mu\psi^*(x)\tilde{D}_\nu\psi(x)\\ &\quad - (m^2 + \xi R)\psi^*(x)\psi(x) - \frac{1}{2}F_{\mu\nu}(x)F^{\mu\nu}(x)\Bigg]\end{aligned} \tag{23}$$

will, by variation with respect to the matter fields $\psi^*(x)$ and $\psi(x)$, give the Euler–Lagrange equations, which give the Klein–Gordon equation. For $\psi(x)$

$$\delta S = \int \sqrt{g}\, d^4x \,[g^{\mu\nu}(\delta(D_\mu\psi^*) D_\nu\psi) - (m^2 + \xi R)\psi(\delta\psi^*)] = 0 \quad (24)$$

Using that $\delta(D_\mu\psi) \equiv D_\mu(\delta\psi)$ and performing the integrations by parts, we get

$$\int d^4x \;\sqrt{g}\,[g^{\mu\nu} D_\mu(\delta\psi^*) D_\nu\psi]$$

$$= \left[\delta\psi^* \frac{\partial\mathcal{L}}{\partial(D_\mu\psi^*)}\right]_{-\infty}^{+\infty} - \int d^4x \;\sqrt{g}\,\frac{1}{\sqrt{g}}\,(\delta\psi^*) D_\mu \frac{\partial\mathcal{L}}{\partial(D_\mu\psi^*)} \quad (25)$$

Note that since the measure in the action integral is $\sqrt{g}\, d^4x$, we have had to add the term $\sqrt{g}(1/\sqrt{g})$ in the last term! This finally gives

$$\delta S = \int d^4x \;\sqrt{g}\left[\frac{1}{2}\frac{1}{\sqrt{g}} D_\mu(-\sqrt{g} g^{\mu\nu} D_\nu\psi(x)) - (m^2 + \xi R)\psi(x)\right]\delta\psi^*(x = 0 \quad (26)$$

or

$$\frac{1}{\sqrt{g}} D_\mu(\sqrt{g}\, g^{\mu\nu} D_\nu\psi(x)) + (m^2 + \xi R)\psi(x) = 0 \quad (27)$$

3.1. The Smeared Position Basis

The “smeared” position basis is introduced for two reasons. We cannot expand our fields on the plane wave basis since we are in curved space. The problem is that we have no well-defined global vacuum, and as a result no “ground states” are globally defined. Since we are interested in localizing our fields within a certain range of probability, we turn our attention to characteristic or indicator functions for the chronological sets (defined in Section 1), that is,

$$\langle x | 1_{U_i}\rangle = 1_{U_i}(x) = \begin{cases} 1 & \text{if} \quad x^\mu \in U_i \\ 0 & \text{if} \quad x^\mu \notin U_i \end{cases}$$

This basis is obviously “over”complete, so we make no assumption of its orthonormality. The indicator functions are discontinuous, but from distribution theory[(12)] we can take functions $\{\chi_{U_i}\}$ defined as[6]

[6] The proof can be found in Rudin’s book on functional analysis[(12)] and is a result of the Hahn–Banach theorem. There, however, $M = \mathbb{R}^n$. Generalizing it to curved spacetimes is done by “pulling” the open subsets of $\mathbb{R}^4$ back to our manifold M using the chart maps φ.

$$\chi_{U_i}(x) = \begin{cases} 1 & \text{if} \quad r_- \leq |x| \leq r_+ \\ ]0;\ 1] & \text{if} \quad R_- \leq |x| \leq r_- \quad \text{or} \quad r_+ \leq |x| \leq R_+ \\ 0 & R_- \geq |x| \geq R_+ \end{cases} \tag{28}$$

As mentioned, we have no orthonormality. What we do have is

$$\langle \chi_{U_i} | \chi_{U_j} \rangle = \int_{U_i \cap U_j} \sqrt{-g}\, d^4x = \mathrm{Vol}(U \cap V)$$

which is equal to zero if $U_i \cap U_j = \emptyset$ and for $i = j$ equal to the volume of the set.

We now simply expand the matter fields $\hat{\psi}(x) \in \mathcal{B}(\mathcal{H})$ in the usual manner:

$$\begin{aligned} \hat{\psi}(x) &= \sum_j \chi_{U_j} \langle \chi_{U_j} | \psi(x) \rangle \\ &= \sum_j \hat{b}_{U_j}(t) \chi_{U_j}(\vec{x}) \end{aligned} \tag{29}$$

Similarly we get for $\hat{\psi}^*(x)$ that

$$\hat{\psi}^*(x)\rangle = \sum_i \hat{b}^\dagger_{U_i}(t) \chi_{U_i}(\vec{x}) \tag{30}$$

The operators $\hat{b}^\dagger_{U_i}(t)[\hat{b}_{U_j}(t)]$ create (annihilate) particles in the sets $U_j(U_i)$ at time t. Their commutator is

$$[b_{\chi_{U_i}}(t), b^\dagger_{\chi_{U_j}}(t)] = \langle \chi_{U_i} | \chi_{U_j} \rangle \tag{31}$$

The electromagnetic field $A^q_\mu(x)$ is also expanded on the basis $\{\chi_{U_i}\}$ as

$$\begin{aligned} \hat{A}_\mu(x) &= \sum_l [\varepsilon^\kappa_\mu\, \hat{a}_{\kappa U_l}(t) \chi_{U_l}(\vec{x}) + \varepsilon^{\kappa *}_\mu \hat{a}^\dagger_{\kappa U_l}(t) \bar{\chi}_{U_l}(\vec{x})] \\ &= \sum_l [\hat{a}_{\mu U_l}(t) \chi_{U_l}(\vec{x}) + \hat{a}^*_{\mu U_l} \chi_{U_l}(\vec{x})] \end{aligned} \tag{32}$$

In the following we use the notation

$$b_i \equiv \hat{b}_{U_i}(t),$$

$$b_i^\dagger \equiv \hat{b}^\dagger_{U_i}(t),$$

$$\chi_i \equiv \chi_{U_i},$$

$$a_l^\mu \equiv \varepsilon^\mu \hat{a}_l(t)$$

3.2. The Legendre Transformation

To be able to use the decoherent histories which will be introduced in Section 5, we first make the transformation from configuration space to phase space. In going from the Lagrangian picture to the Hamiltonian picture, we should specify the fields only on 3-surfaces. Let $U_i \subset M$ be a chronological set from the topology $\tau(\mathcal{U})$ and σ a space like hypersurface. Then let K_i be defined as

$$K_i = U_i \cap \sigma \tag{33}$$

Due to the definition of U_i, K_i will be a 3-sphere contained in both σ and U_i. If $\chi_{U_i} = U(\sigma_i, \sigma_j)\chi_{U_j}$, then clearly U_i must belong to the future domain of dependence of U_j. It will be easier to work, however, with K_i as unit balls since then we can use

$$Vol(K_i) = 1 \qquad \text{and} \qquad K_i \subset K_j \Rightarrow K_i = K_j \tag{34}$$

Later we will let K_ω denote the entire causality-violating area, while $K_{\omega+}$ will denote the set containing the wormhole mouth in the present and $K_{\omega-}$ denotes the set containing the wormhole mouth in the past. Obviously $K_{\omega+} \subset K_\omega$ and $K_{\omega-} \subset K_\omega$.

We previously established the form of the Lagrangian as

$$\begin{aligned}\mathcal{L}(x) = \sqrt{g}\Bigg[& g^{\mu\nu}(D_\mu\psi^*(x)D_\nu\psi(x)) + ie\partial_\mu\psi^*(x)A_0^{cl}(x)\psi(x)g^{\mu 0} \\ & -ieA_0^{cl}(x)\psi^*(x)\partial_\nu\psi(x)g^{0\nu} - (m^2 + \chi R)\psi^*(x)\psi(x) \\ & -\frac{1}{4}(2F_{0i}F^{0i} - F_{\mu\nu}F^{\mu\nu})\Bigg]\end{aligned} \tag{35}$$

The canonical momentas are then found as usual by

$$\begin{aligned}\pi(x) &= \frac{\partial\mathcal{L}}{\partial\dot{\psi}(x)} = \frac{\partial\mathcal{L}}{\partial\,\partial_0\psi(x)} \\ &= \sqrt{g}[\partial_0\psi^*(x) - ieA_0(x)\psi^*(x)\end{aligned} \tag{36}$$

$$\begin{aligned}\pi^*(x) &= \frac{\partial\mathcal{L}}{\partial\dot{\psi}^*(x)} = \frac{\partial\mathcal{L}}{\partial\partial_0\psi(x)} \\ &= \sqrt{g}[\partial_0\psi(x) + ieA_0(x)\psi(x)]\end{aligned} \tag{37}$$

The canonical momenta for the A_μ field are found in a similar manner:

$$\pi_\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_0 A^\mu)} = \sqrt{g}F_{0\mu} \tag{38}$$

Rearranging the above equations gives

$$\dot{\psi}^*(x) = \frac{1}{\sqrt{g}} g_{00}\pi(x) - ieA_0(x)\psi^*(x) \tag{39}$$

$$\dot{\psi}(x) = \frac{1}{\sqrt{g}} g_{00}\pi^*(x) + ieA_0(x)\psi(x) \tag{40}$$

$$\dot{A}_\mu(x) = F_{0\mu} + \partial_\mu A_0 = \frac{1}{\sqrt{g}} \pi_\mu + \partial_\mu A_0 \tag{41}$$

The Hamiltonian density is then found as

$$\begin{aligned} \mathcal{H} &= \mathcal{H}^{(0)} + \mathcal{H}^{(1)} \\ &= \sqrt{g}\, 2g^{00}\dot{\psi}^*(x)\dot{\psi}(x) - \frac{1}{\sqrt{g}} \mathcal{L}_\psi \\ &\quad + g^{00}g^{ij}(2F_{0j}\, F_{0i}) - \frac{1}{\sqrt{g}} \mathcal{L}_A \end{aligned} \tag{42}$$

To keep the notation simple, we shall introduce the measure

$$d^4m(x) = \sqrt{g}\, d^4x$$

4. THE DENSITY MATRIX

An elegant and very useful way of representing a state without the phase arbitrariness is to characterize it by a Hermitian matrix ρ, defined for a pure state $|\Psi(x)\rangle$ by

$$\rho = |\Psi(x)\rangle\langle\Psi(x)| \tag{43}$$

which obeys $\mathrm{tr}\rho = 1$. This matrix is called the *density matrix*, and it plays an important role later when we embark on the decoherent–histories approach to quantum mechanics. According to the probability doctrine of quantum mechanics, its knowledge exhausts all that we can find out about the state.[9] If A is an observable for the system, then the expectation value of A is

$$\langle A\rangle = \mathrm{tr}(\rho A) = \mathrm{tr}(A\rho) \tag{44}$$

The action of this matrix is

$$\rho: \quad \mathcal{H} \to \mathcal{H}, \qquad \rho: \quad \mathcal{H}^\dagger \to \mathcal{H}^\dagger \tag{45}$$

Somewhat cryptically ρ can be said to be a *Hilbert-space-valued linear functional on the Hilbert space*, that is,

$$\rho \in \mathcal{H} \otimes \mathcal{H}^\dagger \tag{46}$$

or, since $\mathcal{H}^\dagger = \mathcal{H}$,

$$\rho \in \mathcal{H} \otimes \mathcal{H} \tag{47}$$

The density matrix is also useful in the definition of missing information, entropy, of a system, since we can define this as

$$S = -\mathrm{tr}[\rho \ln \rho] \tag{48}$$

5. DECOHERENCE AND DECOHERENT HISTORIES

Handling quantum mechanics in curved spacetimes requires care. A way out of the problems appears to be a mix of decoherence, decoherent histories, and nonunitary evolution, which I shall present below. Both decoherence and decoherent histories are concerned with the emergence of an approximately classical universe from an underlying quantum one without having to deal with the details of observers, measuring devices, or the collapse of the wave function. But while decoherence, studied in the next section, is concerned with the quick dispersal of phase information representing interference between different states in a superposition among a set of ignored variables that interacts with some followed variables, which is local in time, the decoherent histories studied in Section 5.2 have variables specified on a sequence of spacelike hypersurfaces and are not local in time. The decoherent-histories formalism also tries to supply a quantum mechanical framework for reasoning about the properties of a *closed system*. In that way it is a predictive formulation of quantum mechanics for genuinely closed quantum systems that is sufficiently general to cope with the needs of quantum cosmology.[8] What replaces the notion of measurement is the more general and objective notion of consistency (or the stronger notion of decoherence), which determines which stories may be assigned probabilities.[8] It should also be noted that this approach to quantum mechanics does not contradict the Copenhagen interpretation of quantum mechanics; it is simply the way it is interpreted that is different.

5.1. Decoherence of the Density Matrix

According to the Schrödinger equation, an initially free wave packet such as our matter field for $t \to -\infty$ would spread over time, thereby increasing its size and extending its coherence properties over a larger region in space. The superposition of partial waves thus seems to make it impossible to ask if our charged scalar particle entered the wormhole or not, since it is not localized—or we may be faced with the problem that only "part" of the wavepacket entered the wormhole.

But the spreading of the wavepacket is known to be negligible for large masses or other macroscopic properties. For example, the position of a dust particle becomes "classical" through scattering of a very large number of air molecules and photons acting together like a continually active position monitor (a "measuring device"). Phase relations between different positions are continually destroyed (or rather delocalized into the environment) in this manner.[7]

Superpositions of *macroscopically* different properties (like position) can be shown[7] to disappear from the density matrix describing the system (i.e., $\rho_{nm} = 0$ for $n \neq m$) on short time scales. This is basically what is meant by *decoherence*.

Even objects which are usually regarded as "microscopic" can acquire classical properties very rapidly due to the formation of quantum correlations with their environment. Quantum mechanical interference disappears if, for example, the passage through the slits of an interference experiment is *measured*. In this case the frequencies of events on the detection screen following a certain passage can be counted separately, and must thus simply be added. So interference should therefore disappear when the passage is measured *without anybody ever looking at the result*. This means that the outcome (here, a certain passage) can be assumed to have become a "classical fact" as soon as the measurement has *irreversibly occurred*. That is, properties of quantum objects "come into being" (or "occur") in an irreversible act by their measurement (not their *observation* as such). *Decoherence is therefore expected to counteract interference*.

An initially factorizing state will in general not keep this property if some kind of interaction is present, but will evolve into an entangled one. This evolution leads to a behavior of a *subsystems* density matrix or the *reduced density matrix*, which may be quite different from the properties the system would show in isolation. Giulini *et al.*[7] give a good example of this, which we shall go through here: A "measurement-like process" is one in which a system acts on its environment in a certain way, while the backreaction from the environment on the system is negligible small (for example, when an electron scatters off a heavy atom, the recoil from the heavy atom can be considered negligible). It is the interaction between the system and its environment which is "measurement like," and no collapse is assumed. If the interaction is of the *von Neumann form*

$$H_I = \sum_n \left|n\right\rangle\left\langle n\right| \otimes \hat{A}_n \tag{49}$$

where $\hat{A}_n$ are n-dependent operators acting only in the Hilbert space of the environment and $\left|n\right\rangle$ is an eigenstate of the "observable" measured by this interaction, $\left|n\right\rangle$ will not change during the interaction while the environment acquires "information" about it. That is, the environmental states (denoted by $\left|\Phi_n\right\rangle$) change in an "$\left|n\right\rangle$-dependent" way:

$$\left|n\right\rangle\left|\Phi_0\right\rangle \xrightarrow{t} e^{-iH_It}\left|n\right\rangle\left|\Phi_0\right\rangle = \left|n\right\rangle e^{-i\hat{A}_nt}\left|\Phi_0\right\rangle \equiv \left|n\right\rangle\left|\Phi_n(t)\right\rangle \tag{50}$$

The resulting environmental states $\left|\Phi_n(t)\right\rangle$ are generally called "*pointer positions*," although they do not need to correspond to any states of a measurement devices. They are simply the states of the "rest of the world."

The Schrödinger equation now yields

$$\left(\sum_n c_n\left|n\right\rangle\right)\left|\Phi_0\right\rangle \xrightarrow{t} \sum_n \left(c_n\left|n\right\rangle\left|\Phi_n(t)\right\rangle\right) \tag{51}$$

that is, a correlated state representing a superposition of all "measurement results." The local density matrix (the density matrix of the subsystem) changes accordingly:

$$\rho_S = \sum_{n,m} c_m^* c_n\left|m\right\rangle\left\langle n\right| \xrightarrow{t} \sum_{n,m} c_m^* c_n\left\langle\Phi_m\middle|\Phi_n\right\rangle\left|m\right\rangle\left\langle n\right| \tag{52}$$

The nondiagonal elements of the local density matrix (which in this basis are defined by the interaction) are thereby multiplied by a factor which is given by the overlap of pointer states corresponding to the respective quantum numbers. Diagonal elements are unchanged. If the environmental states are orthogonal,

$$\left\langle\Phi_m\middle|\Phi_n\right\rangle = \delta_{nm} \tag{53}$$

that is, if the environment discriminates among states, the system density matrix becomes diagonal in this basis:

$$\rho_S \to \sum_n \left|c_n\right|^2\left|n\right\rangle\left\langle n\right| \tag{54}$$

During this evolution, the interference terms (the nondiagonal elements) are destroyed locally (they are delocalized) in this basis, which is defined by the interaction Hamiltonian. *This means that the phase relations characterizing the superposition become inaccessible for local observations.*

5.2. Decoherent Histories

Spacetime must be foliable by spacelike hypersurfaces before the quantum mechanics of matter fields can be formulated in terms of unitarily evolving state vectors defined on spacelike hypersurfaces.[9]

If this foliation is possible, we can use the time coordinate to globally decompose the (1 + 3)-dimensional Lorentzian metric via the *Arnowitt–Deser–Misner* (ADM) split[13]

$$g_{\mu\nu}(t, \vec{x}) = \begin{pmatrix} -(N^2 - g^{ij}\beta_i\beta_j) & \beta_j \\ \beta_i & g_{ij} \end{pmatrix} \tag{55}$$

The entries in this matrix are

$N(t, \vec{x})$, the lapse function;

$\vec{\beta}(t, \vec{x})$, the shift function; and

$g_{ij}(t, \vec{x})$, the metric of the spacelike hypersurfaces.

N and β describe how the spacelike hypersurfaces are assembled to form spacetime, while g_{ij} naturally describes the three-geometry of space.

In a spacetime with closed timelike curves, this foliation by spacelike hypersurfaces is no longer possible, and we need a more general formulation of our quantum field theory. An elegant formulation has been put forward in a paper by Hartle[(9)] in which he describes a generalized quantum theory whose probabilities consistently obey the rules of probability theory even in the presence of an area containing closed timelike curves.

In the Schrödinger picture, an exhaustive and exclusive set of alternatives (events) defined on a spacelike surface Σ (or at time t when spacetime is foliable) corresponds to a set of idempotent, self-adjoint positive operators, that is, a set of projection operators $\{P_\alpha\}$ with $P_\alpha \in \mathbb{B}(\mathcal{H})$, the algebra of bounded operators on $\mathcal{H}$ satisfying at each moment (on each spacelike hypersurface)

$$\sum_\alpha P_\alpha = 1, \qquad P_\alpha P_\beta = \delta_{\alpha\beta} P_\beta \tag{56}$$

These projections correspond to a "proposition" or "event" and *a piori* act on the Hilbert space for the entire system (e.g., matter field and electromagnetic field), thereby producing an appropriate subspace of this Hilbert space. These subspaces may be states describing "the position of x of the field to lie within the range Δx," "its momentum p to lie within the range Δp," or "its spin to point in the z direction."

On each of the spacelike surfaces $\sigma_1, \ldots, \sigma_n$ corresponding to times $t_1, \ldots, t_n$ when spacetime is foliable, we specify sets of alternatives (events)

$$\{P^1_{\alpha_1}\}, \{P^2_{\alpha_2}\}, \ldots, \{P^n_{\alpha_n}\}$$

so the set $\{P^1_{\alpha_1}\}$ acts on the Hilbert space $\mathcal{H}$ at time t_1, the set $\{P^2_{\alpha_2}\}$ acts on the Hilbert space $\mathcal{H}$ at time t_2, etc.

The formal definition of a history is given by a "time-ordered" sequence of quantum alternatives (events):

$$C_\alpha = P^n_{\alpha_n}(t_n)P^{n-1}_{\alpha_{n-1}}(t_{n-1}) \cdots P^1_{\alpha_1}(t_1) \tag{57}$$

with a "time-ordering"[7] $t_n > t_{n-1} > \cdots > t_1$. The subscript k in $P^k_{\alpha_k}(t_k)$ denotes the set of projectors for the surface Σ_k,[8] and α_k denotes the particular alternative which has been chosen; k could be for example, space projection and α_k could be Δx. So a particular history corresponds to a particular sequence of alternatives $\alpha_1, \ldots, \alpha_n$, and we shall use the shorthand notation $\alpha = (\alpha_1, \ldots, \alpha_n)$.

From simple quantum measuring systems we already know that a certain "coarseness" of probabilities is needed in the description if the behavior of the system is to become approximately classical. A good example is Heisenberg's uncertainty relations, which effectively limit the precision with which position and momentum can be measured at the same time. Coarse-graining can be achieved by splitting the Hilbert space variables into "ignored" or "environment" variables and "followed" or "system" variables. This split is possible if the backreaction from the environment onto the system is negligible. By a coarse-grained *history* we then mean a history for which (1) not all the variables are specified, a coarse-graining I call *first type*, and (2) the variables that are specified are not specified at each and every instant (or on each and every spacelike hypersurface) *or* only with an arbitrary precision, a coarse graining I refer to as *second type*.

A coarse-graining of the "second type" can be achieved by defining partitions of the histories $\{\alpha\}$ into classes $\{\overline{\alpha}\}$. That is, a history belonging to $\{\overline{\alpha}\}$ is a whole set of histories contained in $\{\alpha\}$. We can represent a coarse-grained set of histories by the operators $C_{\overline{\alpha}}$, with

$$C_{\bar{\alpha}} = \sum_{\alpha \in \bar{\alpha}} C_\alpha = \sum_{(\alpha_1, \ldots, \alpha_n) \in \bar{\alpha}} P^n_{\alpha_n}(n_n) \cdots P^1_{\alpha_1}(n_1) \tag{58}$$

The smeared position basis (28) introduced at the end of Section 3 is also an example of the second type of coarse graining. Instead of asking for the exact value of the field for all $m \in M$, we ask for the value on the field on an entire subset of M.

Under what circumstances, then, is it possible to assign probabilities to histories? Since the probability for a single "event" α_k is given by the usual formula,

[7] The "distance" between the spacelike hypersurfaces is not arbitrary. When studying the decoherence of the density matrix through interaction with the environment, a *decoherence time scale* t_{dec} is defined. In a scattering process where k is the wave number, Nv/V the incoming flux, and σ_{eff} is of the order of the total cross section for the scattering, the *localization rate* Λ, which governs the destruction of coherence between different positions, is $\Lambda = k^2 Nv\sigma_{\text{eff}}/V$, and the decoherence time scale over a distance $\Delta x = x - x'$ is $t_{\text{dec}} = 1/\Lambda(\Delta x)^2$. Therefore the "time difference" $\sigma_2 - \sigma_1'$ should be chosen larger than t_{dec}.

[8] Or a time t_k when we can use the time parameter.

$$p_k = \langle P_k \rangle = \mathrm{tr}[P_k^\dagger \rho_0 P_k] \tag{59}$$

with ρ_0 denoting the initial density matrix, the obvious generalisation to a history C_α is

$$p(\alpha) = \mathrm{tr}[C_\alpha^\dagger \rho_0 C_\alpha] \tag{60}$$

But a quantum theory of a closed system does not assign probabilities to every set of coarse-grained histories. A good example is the double-slit experiment. If α denotes the passage through the upper slit and β the passage through the lower slit, we can write a coarse-grained history given by just a sum of two histories:

$$C_{\bar{\alpha}} = C_\alpha + C_\beta \tag{61}$$

The probability for the coarse-grained history $C_{\bar{\alpha}}$, asking for the probability that the particle went through either the upper and lower slit, would then be

$$\begin{aligned} p(C_{\bar{\alpha}}) &= p(C_\alpha) + p(C_\beta) + 2\,\mathrm{Re}\,\mathrm{tr}[C_\alpha^\dagger \rho_0 C_\beta] \\ &\neq p(C_\alpha) + p(C_\beta) \end{aligned} \tag{62}$$

The quantum mechanics of a closed system can only assign probabilities to members of sets of alternative, coarse-grained histories for which there is negligible interference between the individual histories. This can be caused by the system's dynamics or boundary conditions.

Equation (62) motivates the introduction of the *decoherence functional*, which is a functional of two histories α and α', and in a sense "measures the interference":

$$\begin{aligned} \mathcal{D}(\alpha, \alpha') &= \mathrm{tr}[C_\alpha^\dagger \rho_0 C_{\alpha'}] \\ &= \mathrm{tr}[P_{\alpha_n}^n(t_n) \cdots P_{\alpha_1}^1(t_1) \rho_0 P_{\alpha_1'}^1 \cdots P_{\alpha_n'}^n(t_n)] \end{aligned} \tag{63}$$

or, written in the Schrödinger picture, using

$$P_{\alpha k}^k(\sigma_k) = U^\dagger(t_{k-1}, t_k) P_{\alpha 1}^1 U(t_{k-1}, t_k) \tag{64}$$

we have

$$\begin{aligned} \mathcal{D}(\alpha, \alpha') = \mathrm{tr}[&P_{\alpha_n}^n U(t_n, t_{n-1}) P_{\alpha_{n-1}}^{n-1} \\ &\cdots P_{\alpha_1}^1 U(t_1, t_0) \rho_0 P_{\alpha_1'}^1 U(t_0, t_1) \cdots P_{\alpha_{n-1}'}^{n-1} U(t_{n-1}, t_n) P_{\alpha_n'}^n] \end{aligned} \tag{65}$$

The operators given by

$$U(t_k, t_{k-1}) = \exp\left[-i\int dt\, H(t_k, t_{k-1})\right] = \exp\left(-i\int_{t_{k-1}}^{t_k} \mathcal{H}\, d^4x\right) \quad (66)$$

are unitary evolution operators between the spacelike surfaces t_k and t_{k-1}, and H is the Hamiltonian. Then $\mathcal{D}(\alpha, \alpha)$ describes probabilities if

$$\text{Re}\, \mathcal{D}(\alpha, \alpha') = 0, \qquad \forall \alpha \neq \alpha' \quad (67)$$

which is is called *weak decoherence* or *consistency*, that is, there is negligible interference between the alternatives α and the alternatives α'. When a set of histories decoheres weakly, the probability of a history $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n$ is the corresponding diagonal element of the decoherence functional[6]:

$$p(\alpha) = \mathcal{D}(\alpha, \alpha) \quad (68)$$

If Re $\mathcal{D}(\alpha, \alpha') = 0$, $\alpha \neq \alpha'$, then these numbers will obey the sum rule of probability theory. That this is indeed the case is easily seen; that Re $\mathcal{D}(\alpha, \alpha') = 0$ means that the interference terms between the two histories α and α' vanish. Gell-Mann and Hartle[6] imposed a stronger condition which demands that the nondiagonal elements of the *whole* decoherence functional be zero:

$$\mathcal{D}(\alpha, \alpha') = 0, \qquad \forall \alpha \neq \alpha' \quad (69)$$

which is called *medium decoherence*.

If $\mathcal{D}(\alpha, \alpha') = 0$, $\forall \alpha \neq \alpha'$, then, as we saw above, it is possible to ask for probabilities for a certain history. In our case it would be interesting to find a set of decoherent histories for the system (the charged scalar field) that contained both a history describing the field passing through the time machine wormhole and a history describing the field simply passing by the area, without entering the wormhole. If the probabilities for these two histories are not the same, then nature does, in a sense, distinguish between them. If, e.g., the probability for the scalar field to pass through the time machine wormhole is much smaller than the probability to pass around it, then there would seem to be a universal law working against time travel. Unfortunately, these calculations are long and very tedious and are not included in this article.

The following properties of the decoherence functional follow from its definition

$$\mathcal{D}(\alpha, \alpha') = \mathcal{D}^*(\alpha, \alpha') \quad \text{Hermitian} \quad (70)$$

$$\sum_{\alpha,\alpha'} \mathcal{D}(\alpha, \alpha') = \text{tr}\rho_0 = 1 \quad \text{normalized,} \quad (71)$$

$$\mathcal{D}(\alpha, \alpha) \geq 0 \quad \text{positive diagonal elements,} \quad (72)$$

$$\sum_{\alpha} \mathcal{D}(\alpha, \alpha) = 1 \tag{73}$$

$$\mathcal{D}(\bar{\alpha}, \bar{\alpha}') = \sum_{\alpha \in \bar{\alpha}} \sum_{\alpha' \in \bar{\alpha}'} \mathcal{D}(\alpha, \alpha') \quad \text{obeys the principle of superposition} \tag{74}$$

$$|\mathcal{D}(\alpha, \alpha')| \leq \mathcal{D}(\alpha, \alpha)\mathcal{D}(\alpha', \alpha') \tag{75}$$

where the last condition states that there is no interference with a history whose decoherence functional has a vanishing diagonal element.

Decoherence implies the existence of *generalized records*. If (69) is fulfilled, then the states $C_\alpha|\psi\rangle$ are an orthogonal (but generally incomplete) set. Therefore there exists a set of projection operators P_β whose the states $C_\alpha|\psi\rangle$ are eigenstates,

$$P_\beta(C_\alpha|\psi\rangle) = \delta_{\alpha\beta}C_\alpha|\psi\rangle \tag{76}$$

where the histories now consist of a string C_α adjoined by the operator P_β at *any* time after the final time for the chain. The decoherence functional then becomes

$$\begin{aligned} \mathcal{D}(\alpha, \beta; \alpha', \beta') &= \mathrm{tr}[P_\beta C_\alpha|\psi\rangle\langle\psi|C^\dagger_{\alpha'}P_{\beta'}] \\ &= \mathrm{tr}[\delta_{\alpha\beta}C_\alpha|\psi\rangle\langle\psi|C^\dagger_{\alpha'}\delta_{\alpha',\beta'}] = 0 \end{aligned} \tag{77}$$

This means that the joint probability is equal to $p(\alpha, \beta) = \delta_{\alpha,\beta}p(\alpha)$. So medium decoherence implies the existence of a string of alternatives $\beta_1, \ldots, \beta_n$, at some fixed moment in time after t_n, which are perfectly correlated with the string $\alpha_1, \ldots, \alpha_n$ at the sequence of times $t_1, \ldots, t_n$. So, adding P_β to the chain C_α results in a decoherence functional that contains the joint probability correlating the events/alternatives in the original chain C_α with the P_β, which are therefore referred as generalized records (information about the stories $\{\alpha\}$ is "stored" here). It may be that the P_β do not represent records in the usual sense of being constructed from quasiclassical variables accessible to us, but this means that at any time there is complete information somewhere in the universe about the histories $\{\alpha\}$. Thus, medium decoherence implies the existence of a generalized record. This is actually a biimplication, since the converse is also true.[8]

5.3. Evolution in Causality-Violating Regions

Generalizing the form of the decoherence functional (65) generalizes Hamiltonian quantum mechanics. Let us turn to the wormhole-time machine spacetime and assume that the nonchronal *NC* region is bounded. This *NC* area could be thought of as an intersection between the worldlines of the two wormhole mouths and the future lightcone of the scalar particle from a certain point in time and forward. Then there exist initial and final regions of spacetime before and after the nonchronal region in which familiar alternatives of the spatial field configurations can be defined on spacelike surfaces, and we can study transition probabilities between these alternatives.

Let $NC = K_w$ and restrict attention to alternatives defined on spacelike surfaces "before" NC, a region denoted $\mathcal{IN}(NC)$, or in the region $\mathcal{FN}(NC)$ "after" the nonchronal region. Now suppose the evolution between a spacelike surface σ_- before NC and a spacelike surface σ_+ after NC is not described by a unitary matrix U, but by a nonunitary matrix X. If we replace U with X in the decoherence functional (65), it will no longer satisfy the first four requirements in (70). But the following generalization will:

$$\begin{aligned}\mathcal{D}(\alpha, \alpha') = N \operatorname{tr}[(P^n_{\alpha_n} U(\sigma_n, \sigma_{n-1}) \cdots P^{k+1}_{\alpha_{k+1}} U(\sigma_{k+1}, \sigma_+) X U(\sigma_-, \sigma_k) P^k_{\alpha_k} \\ \times \cdots U(\sigma_2, \sigma_1) P^1_{\alpha_1} U(\sigma_1, \sigma_0) \rho_0 U(\sigma_0, \sigma_1) P^1_{\alpha'_1} U(\sigma_1, \sigma_2) \\ \times \cdots P^k_{\alpha'_k} U(\sigma_k, \sigma_-) X^\dagger U(\sigma_+, \sigma_{k+1}) P^{k+1}_{\alpha_{k+1}} \cdots U(\sigma_{n-1} \cdot \sigma_n) P^n_{\alpha'_n}]\end{aligned} \tag{78}$$

where

$$N = (\operatorname{tr}(X \rho_0 X^\dagger))^{-1} \tag{79}$$

and we now use σ_k instead of t_k, due to the nontriviality of spacetime. The surfaces $\sigma_1, \ldots, \sigma_k$ all lie before σ_-, that is, in the region $\mathcal{IN}(NC)$, while $\sigma_{k+1}, \ldots, \sigma_n$ lie after σ_+ in $\mathcal{FN}(NC)$. The action of the operator X is simply to evolve from the surface σ_- to the surface σ_+.

The decoherence functional (78) defines a quantum theory that reduces to the usual one (65) when the evolution is unitary, but generalizes it when it is not.[9] Its advantages are that it does not violate the probability sum rule and there is no hypersurface dependence of local probabilities.

So decoherence is concerned with decoherence of a reduced density matrix for the system, while the decoherent-histories approach studies the decoherence of histories of alternatives/propositions/events. As shown by Giolini *et al.*,[7] both can be used under certain circumstances, and this will prove to be very helpful in the calculations. In the Schrödinger picture, the density matrix $\rho(t_1)$ may be written

$$\rho(t_1) = U(t_i, t_1)\rho(t_i)U^\dagger(t_i, t_1) \equiv K^{t_1}_{t_i}[\rho(t_i)] \tag{80}$$

where K is the evolution operator for the "path-projected" density matrix. We can then write the decoherence functional as

$$\mathcal{D}(\alpha, \alpha') = \mathrm{tr}[P^n_{\alpha_n} K^{t_n}_{t_{n-1}}(\cdots P^1_{\alpha_1} K^{t_1}_{t_i}[\rho(t_i)] P^1_{\alpha'_1} \cdots) P^n_{\alpha_n}] \tag{81}$$

We need to find projection operators which act only on the "system," that is, is of the form $P^k_{\alpha_k} = P^{k(S)}_{\alpha_k} \otimes \mathbb{I}$. This is possible if the correlations between the system and the environment only affect the dynamics of the system instantaneously (as is the case below). We then write $\mathrm{tr} = \mathrm{tr}_\epsilon \mathrm{tr}_S$, where tr_ϵ is the trace over the environmental degrees of freedom and tr_S the trace over the system, and see if we can write

$$\mathcal{D}(\alpha, \alpha') = \mathrm{tr}_S[P^{n(S)}_{\alpha_n} \tilde{K}^{t_n}_{t_{n-1}}(\cdots P^{1(S)}_{\alpha_1}[\rho_{\mathrm{red}}(t_1)] P^{1(S)}_{\alpha'_1} \cdots) P^{n(S)}_{\alpha'_n}] \tag{82}$$

where $\rho_{\mathrm{red}} = \mathrm{tr}_\epsilon(K^{t_1}_{t_i}[\rho(t_i)])$ is the reduced density matrix, and $\tilde{K}$ now acts on the system alone. This is possible if we can move the trace over the environmental degrees of freedom through all the intermediate terms up to the initial density matrix, which is exactly the case if the correlations only affect the dynamics of the system instantaneously, as is the case treated below.

We can then make decoherent histories if the projection operator $P^{k(S)}_{\alpha_k}$ at time t_k projects on the instantaneous eigenstates of the *path-projected reduced density matrix*:

$$\rho_S(t_k) \equiv \tilde{K}^{t_k}_{t_{k-1}}[P^{k-1(S)}_{\alpha_{k-1}} \tilde{K}^{t_{k-1}}_{t_{k-2}}(\cdots \rho_{\mathrm{red}}(t_1)) \cdots P^{k-1(S)}_{\alpha'_{k-1}}] \tag{83}$$

for all $k = 1, \ldots, n$, because then the projectors commute with this density matrix and therefore they only act on the RHS of $\rho_{\mathrm{red}}(t_1)$ on $P^{k'}_{\alpha_k}$ to yield $\delta_{kk'}$ [or, in our case, $\mathrm{Vol}(K_k \cap K_{k'})$].

We can then write the generalized decoherence functional (78) as [omitting the index (S)]

$$\mathcal{D}(\alpha, \alpha') = N\,\mathrm{tr}[(P^n_{\alpha_n} \tilde{K}^{\sigma_n}_{\sigma_{n-1}}(\cdots P^{k+1}_{\alpha_{k+1}} \tilde{K}^{\sigma_{k+1}}_{\sigma_+} \tilde{X}^{\sigma_+}_{\sigma_-} \tilde{K}^{\sigma_-}_{\sigma_k}(\cdots P^1_{\alpha_1}[\rho_{\mathrm{red}}(t_1)] P^1_{\alpha'_1}) \times \cdots) P^k_{\alpha'_k}) P^{k+1}_{\alpha'_{k+1}} \cdots) P^n_{\alpha'_n}] \tag{84}$$

where $\tilde{X}^{\sigma_+}_{\sigma_-}[\rho_S(\sigma_-)] = X(\sigma_-, \sigma_+)\rho_S(\sigma_-)X^\dagger(\sigma_-, \sigma_+)$.

We can summarize the above by saying that to achieve probabilities for the "paths" of the matter field we must go through the following steps:

1. First we must investigate if a split of the Hilbert space for the *whole* system allows a division into "system" variables and "environment" variables at the initial time, that is, we see if can we write $\rho(t_i) = \rho_S(t_i)\rho_\varepsilon(t_i)$.

2. We then find appropriate projection operators (a choice which *is* made by hand in the theoretical description of a given process), and possibly collect them into classes to achieve coarse-graining.

3. Then we investigate if it is possible to move the trace over the environmental degrees of freedom all the way through to the initial density matrix. If this is possible, the trace should be performed to get a reduced

density matrix for the system alone, which, as shown above, makes finding the consistent histories considerably easier, and, as will be shown below, will help us localize the matter field before it enters the nonchronal region.

4. Then we find the form of the nonunitary evolution operator so it adequately describes the evolution through the *NC* area.

5. Finally, study if $\mathfrak{D}(\alpha, \alpha') = 0, \forall \alpha \neq \alpha'$, in the set of variables we have chosen. If this is the case, we can ask for the probability for the field to propagate through the time machine wormhole and compare it to the probability for the field to pass by the *NC* area.

6. If it is possible to find two decoherent histories, one describing the field propagating through the time machine wormhole and one where the field goes around the wormhole mouth in the present, it is possible to compare the two probabilities.

Section 6 studies the first two steps mentioned above, while the third and fourth steps are treated in Section 7, and the fifth step discussed. As already mentioned the fifth step involves calculations beyond the scope of this article, but will be the subject of future work.

6. DECOHERING THE MATTER FIELD

We achieve coarse-graining of the "first type" by letting the electromagnetic field degrees of freedom play the role of the "environment" and the matter field be the "system," as described above. That is, we write the Hamiltonian for the full system as

$$H = H_{\psi,0} + H_{A,0} + H_I \tag{85}$$

To get the reduced density matrix for the matter fields involves tracing over the electromagnetic field degrees of freedom. This can be accomplished using the *closed path time formalism*[3] and was done by Diósi in his article on density matrices in QED.[4]

We cannot directly use Diósi's results, since he uses the energy-momentum basis and this is not globally well defined in curved spacetime. But we can use some of the calculations and observations. Before we proceed let us also note the following: We work with a charged scalar particle approaching a nonchronal region containing a charged wormhole mouth. Therefore we have only one current $j_\mu(y)$ and a static electric potential $A_0^{cl}(x)$. The equations are further simplified by noting that only the zeroth component of A^{cl} enters the calculations. To keep the notation simple we use $A^{cl}(x) = V(x)$ and $A^q = A$. The task is to investigate if the electromagnetic field will carry information about the different smeared position states of the particle into the environment.

If this happens, interference between these smeared position states can no longer be observed at the system itself.

The scalar field couples nonminimally to the curvature of the wormhole through the term $\xi R\psi^*(x)\psi(x)$. Looking at Eq. ( ) and using the forms for the redshift and shape functions (12) and (13), we see that the scalar curvature is

$$R = \frac{2Q^2}{\left|x_{r_0}\right|^4} \tag{86}$$

Since R is seen to go as r^{-4}, it is also obvious that the gravitational effects only affect the dynamics of the scalar particle when it gets very close to the position of the wormhole mouth. The Coulomb potential $V(x)$ goes as $|r|^{-1}$, so we can perform the trace over the environmental degrees of freedom close to the NC region, but still treat it as flat space, since there the electromagnetic potential dominates over any gravitational effects. So when trying go solve $\rho_{\rm red} = {\rm tr}_\epsilon(K_{t_i}^{t_1}[\rho(t_i)])$ the picture is a scalar particle approaching a static charge density in approximately flat space.[9]

First we turn the photon vacuum state in the interaction picture into a vacuum state in the Schrödinger picture:

$$|0_A\rangle_I = e\left\{-i\int[A\hat{D}A + Vj + Aj]\, d^3m(x)\right\}|0_A\rangle_S \tag{87}$$

with $\hat{D}$ denoting the differential operator acting on the A field. Again inserting the form of U into the right-hand side of Eq. ( ), completing the square using (87), and using the notation ( ), we now obtain the following expression for the reduced density matrix for the charged scalar particle (discarding terms containing jj or VV):

$$\begin{aligned}\rho_{\rm red}(t) = \hat{T}\exp\Bigg\{\frac{i}{2}\int_{x_0,y_0<t} & d^4m(x)\, d^4m(y)[D^{(F)}(x,y)V_+(x)j_+(y)\\ &+ D^{(\bar{F})}(x,y)V_-(x)j_-(y) - D^{(+)}(x,y)V_+(x)j_-(y)\\ &-D^{(-)}(x,y)V_-(x)j_+(y)]\Bigg\}\rho(t_{in})\end{aligned} \tag{88}$$

Note that we have used $D(x, y)$ instead of $D(x - y)$ since the distance between points is not well defined in curved space.

Again evaluating the super-evolution operator to order e^2, we get

[9] The calculations were attempted in curved space, but it turned out to be an impossible task. However, the calculations are carried out in as general a setting as possible.

$$
\begin{aligned}
\rho(\infty) - \rho(-\infty) = \frac{i}{2}\int & dm(x)\, dm(y)[D^{(F)}(x, y)T(V(x)j(y))\rho(-\infty) \\
& + D^{(\hat{F})}(x, y)\bar{T}(V(x)\bar{j}(y))\rho(-\infty) \\
& - D^{(+)}(x, y)V(x)\rho(-\infty)j(y) \\
& - D^{(-)}(x, y)\bar{j}(y)\rho(-\infty)V(x)]
\end{aligned} \tag{89}
$$

The next step is to evaluate this expression in the localization basis $\{\chi_{K_i}\}$, so we can study the components of the reduced density matrix. To keep the notation simple, we use $|n\rangle = |\chi_{K_n}\rangle$. So, evaluating the expression between states $\langle n|$ and $|m\rangle$, we and are led to

$$
\begin{aligned}
&\rho(\infty)_{nm} - \rho(-\infty)_{nm} \\
&= \frac{i}{2}\int dm(x)\, dm(y) \\
&\quad \times \Bigg[\sum_{r,s,t} \langle n|D^{(F)}(x, y)|r\rangle\langle r|V(x)|s\rangle\langle s|j(y)|t\rangle\langle t|\rho|m\rangle \\
&\quad + \langle n|D^{(\bar{F})}(x, y)|r\rangle\langle r|\bar{j}(y)|s\rangle\langle s|V(x)|t\rangle\langle t|\rho|m\rangle \\
&\quad - \langle n|D^{(+)}(x, y)|r\rangle\langle r|V(x)|s\rangle\langle s|\rho|t\rangle\langle t|j\ (y)|m\rangle \\
&\quad - \langle n|D^{(-)}(x, y)|r\rangle\langle r|\bar{j}(y)|s\rangle\langle s|\rho|t\rangle\langle t|V(x)|m\rangle \Bigg] \\
&= \frac{i}{2}\int dm(x)\, dm(y)\Bigg[\sum_{r,s,t} D^{(F)}_{nr}(x, y)V_{rs}(x)j_{st}(y)\rho_{tm}(-\infty) \\
&\quad + D^{(\bar{F})}_{nr}(x, y)V_{st}(x)\bar{j}_{rs}(y)\rho_{tm}(-\infty) \\
&\quad - D^{(-)}_{nr}(x, y)V_{rs}(x)j_{tm}(y)\rho_{st}(-\infty) \\
&\quad - D^{(-)}_{nr}(x, y)V_{tm}(x)\bar{j}_{rs}(y)\rho_{st}(-\infty) \Bigg]
\end{aligned} \tag{90}
$$

First we evaluate the photon propagators. For $D^{(F)}$ and $D^{(\bar{F})}$ we have

$$
D^{(F)}(x, y) = i[\theta(x, y)\langle 0_A|A_0(x)A_0(y)|0_A\rangle + \theta(y, x)\langle 0_A|A_0(y)A_0(x)|0_A\rangle] \tag{91}
$$

$$
D^{(\bar{F})}(x, y) = i[\bar{\theta}(y, x)\langle 0_A|A_0(x)A_0(y)|0_A\rangle + \bar{\theta}(x, y)\langle 0_A A_0(y)A_0(x)|0_A\rangle] \tag{92}
$$

where

$$\theta(x, y) = \begin{cases} 1 & \text{if } x_0 > y_0 \text{ (that is, } y \text{ takes place before } x) \\ 0 & \text{otherwise} \end{cases} \tag{93}$$

and $\bar{\theta} = -\theta$. Using Eq. (32), we can write the A field as

$$\hat{A}_0(x) = \sum_i [\hat{\alpha}_{0,i}(x_0)\chi_i(\vec{x}) + \hat{a}^\dagger_{0,i}(x_0)\chi_i(\vec{x})] \tag{94}$$

Therefore we get

$$\begin{aligned} D^{(F)}(x, y) &= i[\theta(x, y)\langle 0_A | A_0(x)A_0(y) | 0_A\rangle + \theta(y, x)\langle 0_A | A_0(y)A_0(x) | 0_A\rangle] \\ &= i \sum_{i,j} \langle 0_A | a_{0,i}(x_0)a^\dagger_{0,j}(y_0)\chi_i(\vec{x})\chi_j(\vec{y}) \\ &\quad + a_{0,j}(y_0)a^\dagger_{0,i}(x_0)\chi_j(\vec{y})\chi_i(\vec{x}) | 0_A\rangle \end{aligned} \tag{95}$$

To study this, we need to define the commutator relations between $a_{0,j}(y_0)$ and $a^\dagger_{0,i}(x_0)$. Recalling Eq. (7), it can be seen that the commutator must in a sense be the inner product between the arguments of the operators:

$$[a_{0,i}(x_0);\ a^\dagger_{0,j}(y_0)] = g_{00}\Delta(x_0, y_0) \int_{K_i \cap D^+(K_j)} d^3m(x) \tag{96}$$

where $\Delta(x_0, y_0)$ is a smeared Dirac delta distribution, which, properly renormalized, is 1 for $x_0 = y_0$ and goes to 0 when x_0 and y_0 are very far apart, and $D^+(K_j)$ is the future domain of dependence. Then we get [where $\tilde{\mathrm{V}}\mathrm{ol}(K_i) = 3\ \mathrm{Vol}(K_i)/4\pi = 1$, unit balls]

$$[a_{0,i}(x_0);\ a^\dagger_{0,j}(y_0)] = g_{00} \begin{cases} 1, & i = j, \quad x_0 = y_0 \\ \tilde{\mathrm{V}}\mathrm{ol}(K_i \cap D^+(K_j)) & i \neq j, \quad x_0 = y_0 \\ \Delta(x_0, y_0)\tilde{\mathrm{V}}\mathrm{ol}(K_i \cap D^+(K_j)) & i \neq j, \quad x_0 \neq y_0 \\ \Delta(x_0, y_0) & i = j, \quad x_0 \neq y_0 \end{cases} \tag{97}$$

Thus for $i = j$ and $x_0 = y_0$ the commutator is equal to 1, so $\mathrm{Vol}(K_i \cap D^+(K_j))$ is the generalized version of $\delta^3(\vec{x} - \vec{y})$. With the commutator (97) we can write the propagators as

$$\begin{aligned} D^{(F)}(x, y) = i \sum_{i,j} \chi_i(\vec{x})\chi_j(\vec{y})(&\Delta(x_0, y_0)\ \mathrm{Vol}(K_i \cap D^+(K_j)) \\ &+ \Delta(y_0, x_0)\ \mathrm{Vol}(K_j \cap D^+(K_i))) \end{aligned} \tag{98}$$

For $D^{(\bar{F})}(x, y)$, we recall that it is defined on the negative time branch (∞) to $(-\infty)$. So $\theta(y, x) = 1$ when $y_0 < x_0$, and thus except for a sign we get the same result as for $D^{(F)}(x, y)$:

$$D^{(\bar{F})}(x, y) = -i \sum_{i,j} \chi_j(\vec{y})\chi_i(\vec{x})(\Delta(y_0, x_0)\ \mathrm{Vol}(K_j \cap D^+(K_i))$$
$$+ \Delta(x_0, y_0)\ \mathrm{Vol}(K_i \cap D^+(K_j))) \tag{99}$$

The components of these propagators then have the form

$$D^{(F)}_{nr} = i \sum_{i,j} \chi_i(\vec{x})\chi_j(\vec{y})(\Delta(x_0, y_0)\ \mathrm{Vol}(K_i \cap D^+(K_j) \cap K_n \cap K_r)$$
$$+ \Delta(y_0, x_0)\ \mathrm{Vol}(K_j \cap D^+(K_i) \cap K_n \cap K_r)) \tag{100}$$

$D^{(\pm)}$ can be evaluated using eqs. ( ) and ( ) and Fourier-transforming them back to x-space:

$$D^{(-)}(x, y) = i\langle 0_A | A(x)A(y) | 0_A\rangle$$
$$= -\frac{1}{(2\pi)^4}\int dp\ e^{-ip(x-y)}[2\pi i\theta(p_0)\delta(p^2)]$$

which tells us that $p_0 = \pm|\vec{p}|$ since $\delta(p^2) = \delta(p_0^2 - \vec{p}^2)$, where we must choose $p_0 = +|\vec{p}|$. So we get

$$-\frac{i}{(2\pi)^3}\int dp_0\ d\vec{p}\, e^{-ip(x-y)}[\theta(p_0)\delta(p_0^2 - |\vec{p}|^2)]e^{-i(p_0(x_0-y_0)-|\vec{p}||\vec{x}-\vec{y}|)}$$
$$= -\frac{i}{(2\pi)^3}\int d\vec{p}\, e^{-i|\vec{p}|(x_0-y_0)+i\vec{p}\cdot(\vec{x}-\vec{y})}$$
$$= -\frac{i}{(2\pi)^3}\int d|\vec{p}||\vec{p}|^2\ d\cos\theta_p d\varphi_p\ e^{-i|\vec{p}|(x_0-y_0)+i|\vec{p}||\vec{x}-\vec{y}|\cos\theta_p}$$
$$= -\frac{i}{(2\pi)^2}\int d|\vec{p}||\vec{p}|^2\ d\cos\theta_p\ e^{-i|\vec{p}|(x_0-y_0)+i|\vec{p}||\vec{x}-\vec{y}|\cos\theta_p} \tag{101}$$

The cosine integration can easily be performed:

$$\int_{-1}^{1} e^{-i|\vec{p}|((x_0-y_0)-|\vec{x}-\vec{y}|\cos\theta_p)}\ d\cos\theta_p$$
$$= \frac{1}{i|\vec{p}|\cdot|\vec{x}-\vec{y}|}(e^{i|\vec{p}||\vec{x}-\vec{y}|} - e^{-i|\vec{p}||\vec{x}-\vec{y}|}) \tag{102}$$

Inserting this in Eq. (102) and remembering that $|\vec{p}| > 0$ yields

$$-\frac{i}{(2\pi)^2}\int d|\vec{p}||\vec{p}|^2\ d\cos\theta_p\ e^{-i|\vec{p}|(x_0-y_0)+i|\vec{p}||\vec{x}-\vec{y}|\cos\theta_p}$$
$$= -\frac{1}{(2\pi)^2}\int_0^\infty d|\vec{p}||\vec{p}|^2\ \frac{1}{|\vec{p}|\cdot|\vec{x}-\vec{y}|}e^{-i|\vec{p}|(x_0-y_0)}(e^{i|\vec{p}||\vec{x}-\vec{y}|} - e^{-i|\vec{p}||\vec{x}-\vec{y}|})$$

$$= -\frac{i}{2\pi^2|\vec{x} - \vec{y}|}\int_0^\infty |\vec{p}|e^{-i|\vec{p}|(x_0-y_0)} \sin(|\vec{p}|\cdot|\vec{x} - \vec{y}|)$$

$$= -\frac{i}{2\pi^2|\vec{x} - \vec{y}}\frac{2i|\vec{x} - \vec{y}|(x_0 - y_0)}{(|\vec{x} - \vec{y}| - (x_0 - y_0))^2(|\vec{x} - \vec{y}| + (x_0 - y_0)^2)}$$

$$= \frac{(x_0 - y_0)}{\pi^2(|\vec{x} - \vec{y}| - (x_0 - y_0))^2(|\vec{x} - \vec{y}| + (x_0 - y_0))^2} \tag{103}$$

Since

$$D^{(+)}(x, y) = -\frac{i}{(2\pi)^3}\int dpe^{+ip(y-x)}[\theta(-p_0)\delta(p^2)]$$

$$= -\frac{i}{(2\pi)^3}\int dpe^{-ip(x-y)}[\theta(-p_0)\delta(p^2)] \tag{104}$$

we have again $p_0 = \pm|\vec{p}|$, now choosing $p_0 = -|\vec{p}|$, but otherwise the calculations are identical to the above except for a sign, so we get

$$D^{(+)}(x, y) = -\frac{x_0 - y_0}{\pi^2(|\vec{x} - \vec{y}| - (x_0 - y_0))^2(|\vec{x} - \vec{y}| + (x_0 - y_0))^2} \tag{105}$$

These propagators contain knowledge of both the spatial and the temporal separation of the "events" x and y. Since only distance is important, we can evaluate these propagators in $1 + 1$ dimensions. Here the sets $\chi\kappa_s$ around x and $\chi\kappa_t$ around y are simple open sets of $\mathbb{R}$, $\chi\kappa_s =]a_s, b_s[$ and $\chi\kappa_t =]a_t, b_t[$. Thus we are led to

$$D^{(-)}(x, y)_{st} = \langle s|\frac{t_x - t_y}{\pi^2((x - y) - (t_x - t_y))^2((x - y) + (t_x - t_y))^2}|t\rangle \tag{106}$$

$$= \frac{\Delta t}{\pi^2}\int_{a_s}^{b_s} dx\int_{a_t}^{b_t} dy\frac{1}{((x - y) - \Delta t)^2((x - y) + \Delta t)^2} \tag{107}$$

$$= \frac{1}{4\pi^2(\Delta t)^2}\int_{a_s}^{b_s} dx\Bigg(\frac{\Delta t}{x - b_t - \Delta t} + \frac{\Delta t}{x + t - b_t}$$

$$- \log(b_t - \Delta t - x) + \log(b_t + \Delta t - x) - \frac{\Delta t}{x - a_t - \Delta t}$$

$$+ \frac{\Delta t}{x + t - a_t} - \log(a_t - \Delta t - x)$$

$$+ \log(a_t + \Delta t - x)\Bigg) \tag{108}$$

$$= \frac{1}{4\pi^2(\Delta t)^2} (a_t \log(b_s - a_t - \Delta t) + b_s \log(-b_s + a_t - \Delta t)$$
$$- b_t \log(b_s - b_t - \Delta t) - b_s \log(-b_s + b_t - \Delta t)$$
$$- a_t \log(b_s - a_t + \Delta t) - b_s \log(-b_s + a_t + \Delta t)$$
$$+ b_t \log(b_s - b_t + \Delta t) \ + b_s \log(-b_s + b_t + \Delta t) \qquad (109)$$

$$= \frac{1}{4\pi^2(\Delta t)^2} a_s \log\left[\frac{(b_t - a_s - \Delta t)(a_t - a_s + \Delta t)}{(a_t - a_s - \Delta t)(b_t - a_s + \Delta t)}\right]$$
$$+ b_s \log\left[\frac{(a_t - b_s - \Delta t)(b_t - b_s - \Delta t)}{(b_t - b_s - \Delta t)(a_t - b_s + \Delta t)}\right]$$
$$+ a_t \log\left[\frac{(a_s - a_t + \Delta t)(b_s - a_t - \Delta t)}{(a_s - a_t - \Delta t)(b_s - a_t - \Delta t)}\right]$$
$$+ b_t \log\left[\frac{(a_s - b_t - \Delta t)(b_s - b_t + \Delta t)}{(a_s - b_t + \Delta t)(b_s - b_t - \Delta t)}\right] \qquad (110)$$

This will only contribute if $a_s < a_t < \mathrm{b}_s$. If we choose $a_s = 0$ and $a_t - a_s = \delta x$, then $a_t = \delta x$, $b_s = 2$, and $b_t = 2 + \delta x$. We also recall that $\Delta t = (x_0 - y_0)$, which can be negative or positive. If we replace Δt with $|\Delta t|$ in Eq. (110) then we can handle both $D^{(+)}$ and $D^{(-)}$ simultaneously:

$$D_{nr}^{(\pm)}(x, x_0; y, y_0) = \frac{1}{4\pi^2\Delta t^2}\Bigg\{ -2\delta x \ \mathrm{sign}(\Delta t) \log[2 - \delta x + |\Delta t|]$$
$$+ 4 \ \mathrm{sign}(\Delta t) \log\left[\frac{(\delta x + |\Delta t|)(-\delta x + |\Delta t|)}{(-2 - \delta x + |\Delta t|)(-2 - \delta x + |\Delta t|)}\right.$$
$$+ 4\delta x \ \mathrm{sign}(\Delta t) \log(-\delta x + |\Delta t|) \Bigg\} \qquad (111)$$

For $K_r = K_n$ we get

$$D_{nn}^{(\pm)}(x, x_0; x, y_0) = \frac{1}{\pi^2(\Delta t^2 - 4)} + \mathrm{sign}(\Delta t)\frac{\log[2 + |\Delta t|]}{4\pi^2\Delta t^2}$$
$$- \mathrm{sign}(\Delta t)\frac{\log|\Delta t|}{4\pi^2 \ \mathrm{det}^2} \qquad (112)$$

From Eqs. (111) and (112) we see that $D^{(\pm)}$ diverges for $\Delta t \to 0$, which corresponds to moving the charge very close to the electric potential. Thus if we wanted to investigate this we would need some kind of regularization. Another interesting feature is that we must have $\Delta t > \delta x \Rightarrow \Delta t - \delta x > 0$. This is a causality demand on the photon, stating that it is "off-shell" as it travels from the region K_s to the region K_t, thus either $K_t \subset D^+(K_s)$ or $K_s \subset D^+(K_t)$. We see that $D_{ss}^{(\pm)}$ maximizes $D_{st}^{((\pm)}$ ($D_{ss}^{((\pm)}$ is where $\delta x = 0$), thus to a good approximation these propagators are diagonal in the smeared position basis.

Next we evaluate the static electric potential terms of the form

$$
\begin{aligned}
V(x)_{st} &= \langle s|V(x)|t\rangle = \int \chi_s(x)V(x)\chi_t(x)\, d^3m(x) \\
&= eZ \int_{K_s \cap K_t} \frac{1}{|\vec{x} - \vec{x}_{r_0}|}\, d^3m(x) \\
&= eZ \int_{K_s \cap K_t} \frac{r^2}{\sqrt{r^2 + r_0^2 - 2rr_0 \cos\theta}}\, dr\, d\cos\theta\, d\varphi
\end{aligned} \tag{113}
$$

This integral is not easily solved, but the Coulomb potential can be assumed to be highly localized around the wormhole throat x_{r_0}. We can therefore get rid of the x integration in ( ). As a result only sets K_s which are approximately centered around x_{r_0} will contribute to Eq. (113); thus

$$
\int_{K_s \cap K_t} V(x)\, d^3m(x) = \begin{cases} 0 & \text{if } K_s \cap K_t = \emptyset \\ \ll 1 & \text{if } K_s \cap K_t \neq \emptyset \text{ but } s \neq t \\ \text{diverges} & \text{for } K_s = K_t \text{ but } x_{r_0} \notin K_s \\ \text{“1”} & \text{for } K_s = K_t \text{ and } x_{r_0} \in K_s \end{cases} \tag{114}
$$

where "1" simply denotes the maximal value of V_{st}. Thus $V(x)$ is to a good approximation diagonal in this basis, $V_{st} \approx 0$, $s \neq t$. But this also affects the other terms in the RHS of Eq. (90), which to this approximation now yields

$$
\begin{aligned}
\frac{i}{2} \int dm(y) \Bigg[\sum_{r,s,t} & D_{nr}^{(F)}(x, y)V_{rr}(x)j_{rt}(y)\rho_{tm}(-\infty) \\
&+ D_{nr}^{(\bar{F})}(x, y)V_{ss}(x)\bar{j}_{rs}(y)\rho_{sm}(-\infty) \\
&- D_{nr}^{(+)}(x, y)V_{rr}(x)j_{tm}(y)\rho_{rt}(-\infty) \\
&- D_{nr}^{(-)}(x, y)V_{mm}(x)\bar{j}_{rs}(y)\rho_{sm}(-\infty) \Bigg]
\end{aligned} \tag{115}
$$

and using Eq. (112), this can be even further reduced:

$$\frac{i}{2}\int dm(y)\Bigg[\sum_{r,t} D^{(F)}_{nr}(x,y)V_{rr}(x)j_{rt}(y)\rho_{tm}(-\infty)$$
$$+ D^{(\bar{F})}_{nr}(x,y)V_{ss}(x)\bar{j}_{rs}(y)\rho_{sm}(-\infty)$$
$$- D^{(+)}_{nn}(x,y)V_{nn}(x)j_{tm}(y)\rho_{rt}(-\infty)$$
$$- D^{(-)}_{nn}(x,y)V_{mm}(x)\bar{j}_{ns}(y)\rho_{sm}(-\infty)\Bigg] \tag{116}$$

Since the photon propagator term only contained 0-components, the current term $\langle n|j(y)|r\rangle = j_{nr}(y)$ is also rather easy, yielding only

$$i\langle\chi_{K_n}|\pi\psi - \pi^*\psi^*|\chi_{K_r}\rangle \tag{117}$$

The first term is

$$\langle\chi_n|\pi\psi|\chi_{K_r}\rangle$$
$$= i\int dm(y)\, dm(y')\, dm(y''')\, \langle\chi_n|y\rangle\langle y|\pi y'\rangle\langle y'|\psi|y''\rangle\langle y''|\chi_r$$
$$= i\int \chi_n(y)\langle y|\pi|y'\rangle\langle y'|\psi|y''\rangle\chi_r(y'')\, dm(y)\, dm(y')\, dm(y''')$$
$$= i\int \chi_n(y)\pi(y)\delta(y,y')\psi(y')\delta(y',y'')\chi_r(y'')\, dm(y)\, dm(y')\, dm(y'')$$
$$= i\int \chi_n(y)\pi(y)\psi(y)\chi_r(y)\, d^3m(y)$$
$$= i\int \chi_{K_n\cap K_r}(y)\pi(y)\psi(y)\, d^3m(y)$$
$$= i\int \chi_{K_n\cap K_r}\sum_{i,j}(\partial_0 b_j^\dagger(y_0)\chi_j(\vec{y}))b_i(y_0)\chi_i(\vec{y})\, d^3m(y)$$
$$= i\sum_{i,j}(\partial_0 b_j^\dagger(y_0))b_i(y_0)\int \chi_{K_n\cap K_i\cap K_j\cap K_r}(\vec{y})\, d^3m(y) \tag{118}$$

In the second-to-last line we have used that

$$\partial_0(b_j^\dagger\chi_j) = \partial_0\int b_j^\dagger\chi_i\, dm(y) = \int_{K_j}(\partial_0 b_j^\dagger)\, d^3m(y)$$

The second term in Eq. (117) is

$$-i\langle\chi_n|\pi^*\chi^*|\chi_r\rangle$$
$$= i\int \chi_n(y)\pi^*(y)\psi^*(y)\chi_r(y)\, d^3m(y)$$
$$= i\sum_{j,i}(\partial_0 b_i(y_0))b_j^\dagger(y_0)\int \chi_{K_n\cap K_i\cap K_j\cap K_r}(\vec{y})\, d^3m(y)$$

and we shall use the notation

$$i\langle\chi_{K_n}|\pi\psi - \pi^*\psi^*|\chi_{K_r}\rangle$$
$$= i\sum_{ij}(\partial_0 b_j^\dagger(y_0))b_i(y_0)\alpha(n, i, j, r)$$
$$- i\sum_{ji}(\partial_0 b_i(y_0))b_j^\dagger(y_0)\, Vol(K_n\cap K_i\cap K_j\cap K_r)$$
$$= i\sum_{ij}\tilde{j}_{i,j}\, Vol(K_n\cap K_i\cap K_j\cap K_r) = i\tilde{j}_{nr} \tag{120}$$

so that $\tilde{j}_{rt} = -i\tilde{j}_{nr}$.

Inserting Eqs. (104), (118), and (113) into the RHS, of Eq. (90) gives us the following master equation for the various components of the density matrix in the smeared position representation:

$$\dot{\rho}_{nn} = -\sum_t \Gamma_t\rho_{tn} - \sum_r \Gamma_{r\to n}\rho_{nr} + \sum_r \Gamma_{n\to r}\rho_{rn} \tag{121}$$

$$\dot{\rho}_{nm} = -\sum_t \Gamma_t\rho_{tm} - \sum_r \Gamma_{r\to m}\rho_{nr} + \sum_r \Gamma_{m\to r}\rho_{rm} \qquad m \neq n \tag{122}$$

where we have used the following

$$\Gamma_t = \frac{i}{2}\sum_r\int d^3m(y)\,(D_{nr}^{(F)}(x, y)V_{rr}(x)\tilde{j}_{rt}(y) + D_{nr}^{(\bar{F})}(x, y)\tilde{j}_{rt}(y)V_{tt}(x)) \tag{124}$$

$$\Gamma_{r\to n} = \frac{1}{2}\int d^3m(y)\;\; D_{nn}^{(+)}(x, y)V_{nn}(x)\tilde{j}_{rn}(y) \tag{125}$$

$$\Gamma_{n\to r} = \frac{1}{2}\int d^3m(y)\, D_{nn}^{(-)}(x, y)\tilde{j}_{nr}V_{mm}(x) \tag{126}$$

The picture is, that as the scalar particle approaches the charged mouth of the time machine wormhole it interacts with the Coulomb field and *Brems-strahlung* is produced. The phase relations characterizing the superposition of the different smeared position states of the system are carried away into the environment, making it approximately possible to localize the scalar field. Only the terms containing the $D^{(\pm)}$ propagators contribute to the destruction of the off-diagonal elements. As can be seen from Eq. (112) these terms fall off the farther apart x_0 and y_0 are. The current terms also have their maximal value when r is close to m, as can be seen from Eq. (120). So what we have is a "smeared" diagonal as $\tilde{j}_{nr}$ falls off as $\tilde{s}$ ol$(K_r \cap K_n) \ll 1$.[10]

Note. We have been not been very specific about the nonchronal region *NC*. To achieve decoherence through the interaction between the charge of the wormhole and the matter field, we can see that it requires the particle to be "close" to the wormhole. But on the other hand we must still try to achieve this *before* the particle enters the nonchronal region, since we are using the interaction Hamiltonian to achieve decoherence. Two assumption has been made which *do* need further investigation:

- It is possible to move the trace over the environmental degrees of freedom through the nonunitary evolution operator X.
- The closed path time formalism can be used in a spacetime background given here.

The last assumption should be acceptable, since an integral around a nontrivial topological region only gives rise to an extra phase.[(11)] Note also that the evolution operator X is *intrinsically* nonunitary, and not due to loss of information.

7. THE HISTORIES AND THE EFFECTS OF THE TIME MACHINE

Having established the form of the reduced density matrix for the matter field ρ_{red}, we turn our attention to the decoherent histories. Going through these calculations in detail would require more space than is available here, but I shall outline the steps, and conclude this section with a discussion on some of the properties of the time machine wormhole not previously mentioned.

The projection operators to should be of the form $P^k_{\alpha_k} = P^k_{\chi_{K_k}}$. These project onto states of each of the Hilbert spaces in the Fock space $\mathcal{F}(\mathcal{H})$, which are localized in the set K_k.

$$P_{\chi_i}\colon \quad \mathcal{H} \to \mathcal{H}_k = \{\psi \in \mathcal{H} \big| \text{supp } \psi \subset K_k\} \tag{127}$$

Note that these projection operators violate (56), since the basis $\{\chi_{K_i}\}$ is over-

[10] It should also be emphasized that the program of decoherence usually is used to make, e.g., a dust particle's position decohere when it interacts with photons. In the light of the articles by Diosi and Anastopoulos, I found it worth studying if decoherence could be used "the other way round" if the coarse-graining used was sufficiently "coarse."

complete. This will make the decoherence only approximate, but is otherwise allowed.[(7),11]

So on each space like hypersurface we have a set of projections $P_{\chi_{K_i}}$. A given history C_χ would then correspond to threading a certain (smeared) path through spacetime through K_1 at σ_1, K_2 at σ_2, etc., up to a certain set K_- on σ_-, with unitary evolution operators $U(\sigma_k, \sigma_{k-1})$ to generate "movement" from one hypersurface to the next. Then we encounter the nonchronal region.

Now recall the forms of the decoherence functional generalizing Hamiltonian quantum mechanics to nonchronal spacetime regions, with $\rho_{\text{red}}(t_1) = e^L\rho(t_{in})$.

$$\mathcal{D}(\alpha, \alpha') = N\,\text{tr}[(P^n_{\chi_n}\tilde{K}^{\sigma_n}_{\sigma_{n-1}}(\cdots P^{k+1}_{\chi_{k+1}}\tilde{K}^{\sigma_{k+1}}_{\sigma_+}\tilde{X}^{\sigma_+}_{\sigma_-}\tilde{K}^{\sigma_-}_{\sigma_k}(\cdots P^1_{\chi_1}[\rho_{\text{red}}(t_1)]P^{1}_{\chi_1'}) \\ \times \cdots)P^k_{\chi_k'})P^{k+1}_{\chi_{k+1}'}\cdots)P^n_{\chi_n'}] \tag{128}$$

The evolution operators through the regions before and after NC should only act on the system:

$$U(\sigma_n, \sigma_{n-1}) = \exp\left[-i\int_{\sigma_{n-1}<\sigma_n} d^4m(x)\,\mathcal{H}\right] \tag{129}$$

where $\mathcal{H}$ should be of the form $\mathcal{H} = \mathcal{H}_{\psi,0} + \mathcal{H}_I$, where now $\mathcal{H}_I = jDV + VDj$. The ρ_{red} which appears in the decoherence functional is then further reduced every time a new alternative $P^k_{\chi_k}$ is "added" to the chain.

For X we shall make use of an article by Antonsen and Bormann[2] in which a wormhole time-machine Hamiltonian was constructed. Spacetime is divided into regions, region 1 in the present, containing one of the wormhole mouths, region 2 in the past, containing the other mouth, and a number of other regions which are the rest of the universe. A particle entering region 1 at time t then has the probability β of entering the wormhole and appearing

[11] Another possibility was put forward by Anastopoulos,[1] who assumes a cubic lattice with side length L on spatial hypersurfaces of Minkowski spacetime. The centers of each of the cubes are identified with the point coordinates (nL, mL, rL), $n, m, r \in \mathbb{Z}$. A function on the hypersurfaces Σ is introduced as $[\cdot]: \Sigma \to \Sigma$, which takes a point $\vec{x}$ and assigns it at the point $[\vec{x}]$ of the center of the cube of the lattice in which $\vec{x}$ belongs. The paper is especially interesting since he elaborates on the spatial scales and on their relation to the time scale. One demands, that $l_t \ll l_{\text{av}}$, where l_{av} is the scale within which microscopic processes are averaged out (coarse-grained). Finally one has l_{obs}, the scale corresponding to the level of observation, which is determined by the external constraints to the system. Then one finds a coarse-graining operator which assigns to each field in spacetime a number of operators which correspond to the smearing of the field over a lattice cube and are assumed to lie on its center. In this way the set of projections is both exhaustive and exclusive. However, this approach is not very intuitive and I shall proceed using the overexhaustive and nonexclusive operators.

in region 2 at time $t - T$, thus traveling backward in time, while α is the probability for a particle entering region 2 to enter the wormhole there and travel forward in time. Since the mouths are taken to lie deep inside the regions 1 and 2, $\alpha, \beta < 1$. Then the Hamiltonian is written.[2]

$$H = \alpha a_1^\dagger(t + T)a_2(t) + \beta a_2^\dagger(t - T)a_1(t) + g \sum_{t=1}^{N} a_i^\dagger(t)a_i(t) \tag{130}$$

where g counts the number of quanta in the region i, and with a commutator relation for the creation and annihilation operators of the form

$$[a_i(t), a_j^\dagger(t')] = \delta_{ij}\Delta(t - t') + \delta_{i1}\delta_{j2}\Delta(t' - (t + T)) + \delta_{i2}\delta_{j1}\Delta(t' - (t - T)), \quad e, j = 1, 2, \ldots, N \tag{131}$$

Modifications of this allow us to write the nonunitary evolution operator as

$$X(\sigma_+, \sigma_-) = \exp\left[-i \int_{\sigma_- < \sigma_+} d^4m(x)\mathcal{H} + \mathcal{H}_w) \right] \tag{132}$$

with

$$\mathcal{H}_w = \alpha b^\dagger_{K_{w-}}(\sigma_w - \Sigma)b_{K_{w+}}\ \sigma_w)\chi_{w+}(x)\chi_{w-}(x) \tag{133}$$

where α now is the amplitude for the matter particle to enter the wormhole in the region K_{w+}.

The wormhole mouth in the present is deep inside the set K_{w+} on the surface σ_w, while the mouth in the past reside deep inside K_{w-} on the surface $\Delta\sigma_w$ "before" σ_w. The surfaces σ_w and $\Delta\sigma_w$ are "fictitious," since they are merely included to "place" the wormhole mouths; we do not have actual time parameters t and T as in Eq. (130), and we do not expect the surfaces to be spacelike. If the particle enters the set K_{w+} in the surface σ_w, it has the amplitude α to reappear in the region K_{w-} in the surface $\Delta\sigma_w$, i.e., it has moved backward in "time". It is not difficult to generalize this to also include travel forward in time, as can be seen from Eq. (130), from the set K_{w-} to K_{w+}.

The commutator relations between the creation and annihilation operators for the matter field are

$$[b_i, (\sigma); b_j^\dagger(\sigma')] = \Delta(\sigma, \sigma') \int_{K_i \cap D^+(K_j)} d^4m(x) + \Delta(\sigma', \sigma - \Sigma) \int_{K_i \cap K_j \cap D^+(K_{w-}) \cap D^+(K_{w+})} d^4m(x) \tag{134}$$

Outside the nonchronal region only the first term on the RHS of (134) contributes to the commutator, the last term contains the contributions from the time machine.

That X is nonunitary is due to the fact that the Hamiltonian governing the evolution from σ_- to σ_+ is not Hermitian:

$$H_w = \int d^3m(x)\,[\alpha b^\dagger_{K_{w-}}(\sigma_w - \Sigma) b_{K_{w+}}(\sigma_w)\chi_{w+}(x)\chi_{w-}(x)] \tag{135}$$

$$\Rightarrow H^\dagger = \int d^3m(x)\,[\alpha b^\dagger_{K_{w+}}(\sigma_w) b_{K_{w-}}(\sigma_w - \Sigma)\chi_{w+}(x)\chi_{w-}(x)]$$

$$\neq H$$

Looking at Eq. (130), it would seem as though unitarity of X could be achieved if in Eq. (133) we added a term for going *from* the past to the present and assuming $\alpha = \beta$. However, Antonsen and Bormann[2] show that H remains antisymmetric, since they find that there are more quanta exiting the time machine than there are entering it.

The existence of a nonchronal region in the future has a profound effect on the probabilities of alternatives in the present and as a result also the decoherence properties. Assume that spacetime contains a single nonchronal region in the future. We can then ask for the probabilities for a set of alternatives $\{P_\alpha\}$ that all occur *before* the NC region.

If the set of alternatives decohere, we have

$$p(\alpha) = \mathcal{D}(\alpha, \alpha) = N\,\mathrm{tr}[XC_\alpha\rho_0 C\alpha^\dagger X^\dagger] \tag{136}$$

If X had been unitary we could have used the cyclic property of the trace to show that $p(\alpha) = \mathrm{tr}(C_\alpha\rho_0 C^\dagger_\alpha)$, and since the unitary evolutions occur between the alternatives defined on surfaces σ before the final σ_n in the chain, there is no dependence on the future geometry of spacetime. So in a sense unitary evolution implies causality.[9]

But since X is not unitary, the probabilities $p(\alpha)$ will in fact come to depend on the future. If we write $\rho_f = X^\dagger X$ the cyclic property of the trace yields

$$p(\alpha) = N\,\mathrm{tr}[\rho_f C_\alpha \rho_0 C^\dagger_\alpha] \tag{137}$$

where now $N^{-1} = \mathrm{tr}(\rho_f\rho_0)$. That is, we need information about the past ρ_0 *and* the future ρ_f to be able to predict probabilities in the present. Note that it is information about the spacetime geometry of the future, not specific alternatives that happen, that is needed. That probabilities in the present are independent of specific alternatives occurring in the future is guaranteed by the sum rule that follows from decoherence:

Let $\{\alpha\}$ denote a set that we have access to in the present, so that P_α represent the alternatives in the present, and let $\{\beta\}$ be another set in the future, so that P_β denotes the future alternatives. If the set $\{\alpha, \beta\}$ exhibits negligible interference, then it decoheres and we can write for the joint probability

$$p(\alpha, \beta) = N\,\mathrm{tr}[P_\beta X P_\alpha \rho_0 P_\alpha X^\dagger P_\beta] \tag{138}$$

An example could be an *NC* area in our future contained in an impenetrable black box. Observers in the future then have the alternatives of opening the door and letting fields propagate in or leaving the door closed. Since we do not know which β is to be chosen, we must sum over them, and get

$$\begin{aligned} p(\alpha) &= N \sum_\beta \mathrm{tr}[P_\beta X P_\alpha \rho_0 P_\alpha X^\dagger P_\beta] \\ &= N\,\mathrm{tr}[X P_\alpha \rho_0 P \alpha X^\dagger] \end{aligned} \tag{139}$$

that is, probabilities in the present are affected by the existence of the nonchronal region in the future, but not whether the door is open or not (Hartle[9] and J. B. Hartle, personal communication). So our generalized quantum mechanics violates causality in the sense described above, in that we need knowledge of the future. But if this is the case, and if we then restrict our attention to a set of histories on a single spacelike hypersurface σ before the *NC* region, $\{P_\alpha(\sigma)\}$, then the decoherence functional can be written

$$\mathcal{D}(\alpha, \alpha') = N\,\mathrm{tr}[X P_\alpha(\sigma) \rho_0 P_{\alpha'}(\sigma) X^\dagger] \tag{140}$$

and it would automatically decohere if X was unitary, again due to the cyclic properties of the trace. But since X is not unitary, only certain members in the set will. Note, however, that the mechanism of decoherence used in the previous section is essentially local in time, and therefore can be expected to be unaffected by the presence of the *NC* region in the future.[9] So the alternatives that define the quasiclassical domain of the present can still be expected to decohere even when one or more nonchronal regions reside in the future.

7.1. Entropy Change and the Direction of Time

That sets of alternatives on a hypersurface before the *NC* region are affected by the nonunitary evolution operator also affect the missing information of a system, the *entropy*. The idea is to find a good measure of the missing information on a spacelike hypersurface before the nonchronal region and compare it with the missing information on a spacelike hypersurface after the nonchronal region.

Let us again consider a spacetime with only one nonchronal region *NC* to our future and spacelike hypersurfaces σ_- *before* the nonchronal region and σ_+ *after* the nonchronal region. On each of these surfaces we consider decoherent sets of alternatives represented by $\{P_\beta^-\}$ and $\{P_\beta^+\}$, both of which obey (56). Since we assume the sets decohere, the probabilities of these alternatives can be written[9]

$$p(\alpha;\,\sigma_-,\,\{P_\beta^-\}) \approx N\,\mathrm{tr}[P_\alpha^- X^\dagger X\rho] \approx \mathrm{tr}[P_\alpha^-\hat{\rho}] \tag{141}$$

$$p(\alpha;\,\sigma_+,\,\{P_\beta^+\}) \approx N\,\mathrm{tr}[P_\alpha^+ X\rho X^\dagger] \approx \mathrm{tr}[P_\alpha^+\tilde{\rho}] \tag{142}$$

Here $p(\alpha;\,\sigma_-,\,\{P_\beta^-\})\}$ is the probability for the alternative/proposition α from the set $\{P_\beta^-\}$ on the spacelike hypersurface σ_-, and we use the notation

$$\hat{\rho} = \frac{\{\rho,\, X^\dagger X\}}{\mathrm{tr}[X\rho X]} \tag{143}$$

$$\tilde{\rho} = \frac{X\rho X^\dagger}{\mathrm{tr}[X\rho X^\dagger]} \tag{144}$$

with $\{\cdot,\,\cdot\}$ the anticommutator. Next define $S(\sigma;\,\{P_\beta\})$ as the missing information on σ *relative* to the set of alternatives $\{P_\beta\}$ whose probabilities are $\{p_\beta\}$. $S(\sigma;\,\{P_\beta\})$ is the *maximum* of the *entropy functional*:

$$\mathcal{S}(\tilde{\rho}) = -\mathrm{tr}[\tilde{\rho}\,\ln\,\tilde{\rho}] \tag{145}$$

defined over *all* density matrices $\tilde{\rho}$ that reproduce the probabilities $p_\alpha = \mathrm{tr}[P_\alpha\tilde{\rho}]$. If we ask the wrong questions, it is easy to lose information, so to get a good measure of the missing information independent of the "questions" we ask, we should minimize the missing information on σ, $S(\sigma,\,\{P_\beta\})$, relative to *all* decohering sets $\{P_\beta\}$:

$$S(\sigma) = \min_{\substack{\text{decoherent}\\ \{P_\beta\}}} (S(\sigma,\,\{P_\beta\})) = \min_{\substack{\text{decoherent}\\ \{P_\beta\}}} \Big(\max_{\substack{\tilde{\rho}\text{ with}\\ \mathrm{tr}[P_\alpha\tilde{\rho}]=p_\alpha}} \mathcal{S}(\tilde{\rho})\Big) \tag{146}$$

What then, is the relation between the missing information $S(\sigma_-$ and $S(\sigma_+)$ before and after the nonchronal region? Equation (78) shows that for alternatives *after* the nonchronal region the decoherence functional is the same as that for usual quantum mechanics, with the modification that we should use $\tilde{\rho}$ defined in (144). As discussed above, the cyclic property of the trace shows that sets of alternatives defined on a *single* hypersurface after all nonchronal regions always decohere. Therefore

$$S(\sigma_+) = \mathcal{S}(\tilde{\rho}) = \mathcal{S}\left(\frac{X\rho X^\dagger}{\mathrm{tr}[X\rho X^\dagger]}\right) \tag{147}$$

The missing information on the hypersurface σ_- before *NC* is not as easy to calculate, as $S(\sigma_+)$ due to the nontriviality of the strictures of decoherence. But one can show that[(9)]

$$S(\sigma_-) \geq S(\sigma_+) = \mathcal{S}(\tilde{\rho}) = \mathcal{S}\left(\frac{X\rho X^\dagger}{\mathrm{tr}[X\rho X^\dagger]}\right) \tag{148}$$

This means that information can be gained, but not lost, in evolving from the surface σ_- before the *NC* region to a surface σ_+ after all *NC* areas.[12] When recalling the discussion in the previous section, on how a nonchronal region to our future could influence present probabilities and the decoherence of sets of alternatives, the possibility of information *gain* is not so surprising after all. On a given spacelike hypersurface σ_+ to the future of all *NC* areas *any* set of alternatives defined on that surface decohere due to the cyclic property of the trace. But on σ_- only certain sets of alternatives will decohere. So we have more questions (more sets of alternatives) with which to ask questions about the quantum systems on σ_+ than we have on σ_-. The missing information thus decreases corresponding to a decrease in entropy.[13]

Looking back at Eqs. (76) and (77), I realized that if the decoherence in the present is affected by the existence of a nonchronal region to our future, so also is the formation of the generalized records. As we saw in section 5.2, decoherence implies the existence of such records and vice versa. If a set of alternatives decohered, we found the sets $C_\alpha|\psi\rangle$ to be orthogonal

[12] In a spacetime containing several *NC* areas, Eq. (148) only holds when σ_+ is after *all* *NC* areas.[(9)]

[13] In their article on nonunitarity,[(2)] in which they define the time machine Hamiltonian, Eq. (130), Antonsen and Bormann also calculate the entropy $S = -\mathrm{tr}[\rho \ln \rho]$. They conclude, however, that entropy has been created in evolving from time t', before the area containing the time machine probabilities, to time t after. By going through their calculations, however, it becomes obvious that their result is not in contradiction with Eq. (148), but their interpretation is wrong: Up to a normalization constant, ρ is the evolution operator $U(t, t')$ [which we call $X(\sigma_+, \sigma_-)$], so $\rho \ln \rho \approx UH$, where H is the Hamiltonian, Eq. (130). Thus[(2)]

$$S(t) = -\mathrm{tr}[\rho \ln \rho] \approx -\mathrm{tr}[UH] \tag{149}$$

Calculating $\mathrm{tr}[UH]$ and regularizing the result yields

$$\mathrm{tr}[UH] = \frac{1}{72}(t - t')\alpha\beta\Delta(|t - t'| - T) \tag{150}$$

which is positive whenever $\alpha, \beta > 0$, and Antonsen and Bormann conclude that entropy has been generated. *But*, if $\mathrm{tr}[UH] > 0$, then

$$S(t) \approx -\mathrm{tr}[UH] < 0 \tag{151}$$

So if $\mathrm{tr}[UH]_t$ for time t after the time machine area is *greater* than $\mathrm{tr}[UH]_{t'}$ for time t' before, then

$$-\mathrm{tr}[UH]_t < -tr[UH]_{t'} \Rightarrow S(t) < S(t') \tag{152}$$

So it is in fact negative entropy which has been generated, which corresponds to information *gain*.

sets, which would then be eigenstates for a suitable set of projection operators P_β defined after the final time in the chain C_α. We then found the joint probability for the set $\{\alpha\}$ and $\{\beta\}$ to be $p(\alpha, \beta) = \delta_{\alpha\beta}\, p(\alpha)$. But as we have just seen, a nonchronal region to our future affects both probabilities and decoherence properties in the present. Looking at Eq. (77), we see that we have two cases, one where the nonunitary evolution takes place *before* the P_β are added, and one where the nonunitary evolution takes place after this.

Although it looks as though we should simply ask for projection operators to which $XC_\alpha|\psi\rangle$ are eigenstates, we just saw that the set $\{\alpha\}$ may not even decohere any longer. And if the nonunitary evolution takes place *after* we have added the P_β we have also seen that probabilities become dependent on the future geometry of spacetime.

Especially if the records through calculations were expected to be persistent and permanent, how would such a region affect them? Records can be thought of as information somewhere in the universe (be it photographic plates, computer memories, or something else) that an event (or chain of events) occurred. So if there is a nonchronal region in our future it affects the decoherence of alternatives in the present, the entropy, and also the generalized records.

8. CONCLUSION

We have set up a program for working with QED in a nontrivial spacetime. This included introducing the density matrix and the concepts of decoherence of a density matrix and decoherent histories. The latter in particular allowed a generalization of quantum theory to spacetimes with a bounded nonchronal region. The role of the environmental degrees of freedom are played by the electromagnetic field, while the matter field is the system. After studing the effects of the interaction between the charged wormhole mouth and the matter field we reached a smeared diagonal reduced density matrix for the matter field. In other words, an approximate localization occurred.

There are consequences of a nonchronal region in our future. It was seen to affect not only the probabilities for alternatives and histories in the present, but also entropy, decoherence of present alternatives, and in turn also the concept of generalized records. Through this it should in a sense be possible to set up an experiment to "measure" such a nonchronal region in our future by comparing the experimental data with what calculations predict.

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